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Harsanyi's Aggregation Theorem with Incomplete Preferences[†]

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We provide a generalization of Harsanyi's (1955) aggregation theorem to the case of incomplete preferences at the individual and social level. Individuals and society have possibly incomplete expected utility preferences that are represented by sets of expected utility functions. Under Pareto indifference, social preferences are represented through a set of aggregation rules that are utilitarian in a generalized sense. Strengthening Pareto indifference to Pareto preference provides a refinement of the representation. (*JEL* D01, D11, D71)

Harsanyi's (1955) aggregation theorem establishes that when individuals and society have expected utility preferences over lotteries, society's preferences can be represented by a weighted sum of individual utilities as soon as a Pareto indifference condition is satisfied. This celebrated result has become a cornerstone of social choice theory, being a positive aggregation result in a field where impossibility results are the rule, and is viewed by many as a strong argument in favor of utilitarianism.

Harsanyi's result sparked a rich (and on-going) debate about both its formal structure and substantive content (for an overview see, among others, Sen 1986; Weymark 1991; Mongin and d'Aspremont 1998; Fleurbaey and Mongin 2012). An important question, in particular, is how robust the result is to more general preference specifications. Most findings on this issue are negative. For instance, moving from (objective) expected utility preferences over lotteries to subjective expected utility preferences over acts results in an impossibility unless all individuals share the same beliefs (Hylland and Zeckhauser 1979; Hammond 1981; Seidenfeld, Kadane, and Schervish 1989; Mongin 1995; Gilboa, Samet, and Schmeidler 2004; Chambers and Hayashi 2006; Keeney and Nau 2011). This impossibility extends even to the common belief case whenever individual preferences are not necessarily

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neutral toward ambiguity, as are subjective expected utility preferences (Gajdos, Tallon, and Vergnaud 2008).

In this note we take issue with the assumption of complete preferences. There are at least two reasons why one may want to allow for incomplete preferences in social choice theory. First, individuals may sometimes be intrinsically indecisive, i.e., unable to rank alternatives (Aumann 1962; Bewley 1986; Shapley and Baucells 1998; Ok 2002; Dubra, Maccheroni, and Ok 2004; Evren 2008; Ok, Ortoleva, and Riella 2012; Galaabaatar and Karni 2013; Pivato 2013). Second, even if individuals all have complete preferences, these preferences may in practice be only partially identified (Manski 2005, 2011). As we shall see, Paretian aggregation remains possible, when individuals have incomplete expected utility preferences over lotteries, and still has a utilitarian flavor, although in a generalized sense.

I. Statement of the Theorem

Let $\mathcal{X}$ be a finite set of *outcomes* and $\mathcal{P}$ denote the set of all probability distributions (*lotteries*) over $\mathcal{X}$. A *utility function* on $\mathcal{X}$ is an element of $\mathbb{R}^{\mathcal{X}}$. We denote by $e \in \mathbb{R}^{\mathcal{X}}$ the constant utility function $x \mapsto e(x) = 1$.

Shapley and Baucells (1998) and Dubra, Maccheroni, and Ok (2004) show that a (weak) preference relation $\succsim$ over $\mathcal{P}$ satisfies the reflexivity, transitivity, independence, and continuity axioms if and only if it admits an *expected multi-utility representation*, i.e., a convex set $\mathcal{U} \subseteq \mathbb{R}^{\mathcal{X}}$ such that for all $p, q \in \mathcal{P}$,

$$p \succsim q \Leftrightarrow \left[\forall u \in \mathcal{U}, \sum_{x \in \mathcal{X}} p(x)u(x) \geq \sum_{x \in \mathcal{X}} q(x)u(x)\right].$$

These are the standard axioms of the expected utility model (von Neumann and Morgenstern 1944), except that completeness is weakened to reflexivity (and continuity is slightly strengthened). Thus, given these axioms, $\succsim$ is complete if and only if $\mathcal{U}$ can be taken to be a singleton, i.e., a standard *expected utility representation*.

Consider a society made of a finite set $\{1, \ldots, I\}$ of individuals. Each individual $i = 1, \ldots, I$ is endowed with a (weak) preference relation $\succsim_i$ over $\mathcal{P}$ satisfying the above axioms. Society itself is also endowed with a preference relation $\succsim_0$ over $\mathcal{P}$ satisfying these axioms. For all $i = 0, \ldots, I$, denote by $\succ_i$ and $\sim_i$ the asymmetric (strict preference) and symmetric (indifference) parts of $\succsim_i$, respectively. Say that the preference profile $(\succsim_i)_{i=0}^I$ satisfies *Pareto indifference* if for all $p, q \in \mathcal{P}$, $[\forall i = 1, \ldots, I, p \sim_i q] \Rightarrow p \sim_0 q$, and *Pareto preference* if for all $p, q \in \mathcal{P}$, $[\forall i = 1, \ldots, I, p \succsim_i q] \Rightarrow p \succsim_0 q$.

Harsanyi's (1955) aggregation theorem establishes that if $\succsim_i$ is complete and endowed with an expected utility representation $\{u_i\}$ for all $i = 0, \ldots, I$, then (i) $(\succsim_i)_{i=0}^I$ satisfies Pareto indifference if and only if $u_0 = \sum_{i=1}^I \theta_i u_i + \gamma e$ for some $\theta \in \mathbb{R}^I$ and $\gamma \in \mathbb{R}$, and (ii) $(\succsim_i)_{i=0}^I$ satisfies Pareto preference if and only if the same holds with $\theta \in \mathbb{R}_+^I$.[1] Thus, in the expected utility setting, Pareto indifference

[1] See e.g., De Meyer and Mongin (1995) for a rigorous proof in a general setting.

(resp. preference) is necessary and sufficient for the social utility function to consist of a *signed utilitarian* (resp. *utilitarian*) aggregation of individual utility functions.

More generally, let us now endow $\succsim_i$ with an expected multi-utility representation $\mathcal{U}_i$ for all $i = 0, \ldots, I$. This allows for preference incompleteness at both the individual and social level. We then obtain the following generalization of Harsanyi's aggregation theorem. The proof is presented in the Appendix.

THEOREM 1: *Let $\succsim_i$ be a preference relation over $\mathcal{P}$ endowed with an expected multi-utility representation $\mathcal{U}_i$, for all $i = 0, \ldots, I$.*

(*i*) $(\succsim_i)_{i=0}^I$ *satisfies Pareto indifference if and only if*

$$\mathcal{U}_0 = \left\{ \sum_{i=1}^{I} \alpha_i\, u_i - \beta_i\, v_i + \gamma e : \left(\alpha, \beta, \gamma, (u_i, v_i)_{i=1}^I\right) \in \mathcal{L} \right\} \tag{1}$$

for some (α, β)- and $(u_i, v_i)_{i=1}^I$- sectionally convex set $\mathcal{L} \subseteq \mathbb{R}_+^{2I} \times \mathbb{R} \times \prod_{i=1}^I \mathcal{U}_i^2$.[2]

(*ii*) *Assume* $\sum_{i=1}^I \mathrm{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}}$ *is closed.*[3] $(\succsim_i)_{i=0}^I$ *satisfies Pareto preference if and only if*

$$\mathcal{U}_0 = \left\{ \sum_{i=1}^{I} \theta_i u_i + \gamma e \ : \ \left(\theta, \gamma, (u_i)_{i=1}^I\right) \in \mathcal{M} \right\} \tag{2}$$

for some θ- and $(u_i)_{i=1}^I$-sectionally convex set $\mathcal{M} \subseteq \mathbb{R}_+^I \times \mathbb{R} \times \prod_{i=1}^I \mathcal{U}_i$.

Thus, in the expected multi-utility setting, Pareto indifference (resp. preference) is necessary and sufficient for the set of social utility functions to consist of a set of *bi-utilitarian* (resp. *utilitarian*) aggregations of individual utility functions. Bi-utilitarianism aggregates two utility functions u_i and v_i for each individual $i = 1, \ldots, I$, the former with a nonnegative weight α_i and the latter with a non-positive weight $-\beta_i$, thereby generalizing signed utilitarianism (which corresponds to the particular case where $u_i = v_i$ for all $i = 1, \ldots, I$).[4] As in Harsanyi's aggregation theorem, the constants γ in the sets $\mathcal{L}$ and $\mathcal{M}$ do not affect social preferences, so setting them to 0 yields another expected multi-utility representation of $\succsim_0$.

II. Comments

Bi-utilitarianism cannot in general be reduced to signed utilitarianism in part (i) of the theorem, as the following example shows. Let $\mathcal{X} = \{x, y, z, w\}$, $I = 2$,

[2] A set $\mathcal{S} \subseteq \mathcal{S}_1 \times \mathcal{S}_2$ is s_1-sectionally convex if $\{s_2 \in \mathcal{S}_2 : (s_1, s_2) \in \mathcal{S}\}$ is convex for all s_1 in $\mathcal{S}_1$.
[3] cone $(\cdot)$ denotes conical hull and the sum of two sets is the Minkowski sum.
[4] See Danan, Gajdos, and Tallon (2013) for a similar pattern in a multi-profile setting.

$\mathcal{U}_0 = \{u_0\}$, $\mathcal{U}_1 = \{u_1\}$, and $\mathcal{U}_2 = \text{conv}\big(\{u_2^a, u_2^b\}\big)$, where u_0, u_1, u_2^a, u_2^b are as follows:[5]

	u_0	u_1	u_2^a	u_2^b
x	4	1	1	-1
y	1	1	0	0
z	1	0	1	1
w	0	0	0	0

Then for all $p, q \in \mathcal{P}$, $[\forall i = 1, 2, p \sim_i q] \Leftrightarrow p = q$, so $(\succsim_i)_{i=0}^2$ trivially satisfies Pareto indifference (consistently with the theorem, we have $u_0 = u_1 + 2u_2^a - u_2^b$).Yet there exists no $(\theta, \gamma, (u_i)_{i=1}^2) \in \mathbb{R}^2 \times \mathbb{R} \times \prod_{i=1}^2 \mathcal{U}_i$ such that $u_0 = \sum_{i=1}^2 \theta_i\, u_i + \gamma e$.

The closedness assumption in part (ii) is not innocuous in terms of preference: there are profiles $(\succsim_i)_{i=1}^I$ of individual preference relations satisfying the above axioms for which there exists no profile $(\mathcal{U}_i)_{i=1}^I$ of expected multi-utility representations such that $\sum_{i=1}^I \text{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma\in\mathbb{R}}$ is closed. But there are at least two cases where such a $(\mathcal{U}_i)_{i=1}^I$ always exists. The first is when $\succsim_i$ satisfies an additional *finiteness* axiom for all $i = 1, \ldots, I$ (Dubra and Ok 2002). The second is when $(\succsim_i)_{i=1}^I$ satisfies a *minimal agreement* condition. When the closedness assumption is not satisfied, $\mathcal{U}_0$ can only be shown to be included in the closure of the set in the right-hand side of (2) for some $\mathcal{M}$. Details are provided in the Appendix.

As in Harsanyi's aggregation theorem, individual weights are not unique in (1) and (2). Non-uniqueness is more severe when individual preferences are incomplete because the way society selects individual utility functions out of the individual expected multi-utility representations is itself not unique. That is to say, even if $\mathcal{U}_i$ is fixed for all $i = 1, \ldots, I$ and the minimal agreement condition holds, it may be the case that $\sum_{i=1}^I \theta_i u_i + \gamma e = \sum_{i=1}^I \theta_i' u_i' + \gamma' e$ for some $(\theta, \gamma, (u_i)_{i=1}^I) \neq (\theta', \gamma', (u_i')_{i=1}^I) \in \mathbb{R}_+^I \times \mathbb{R} \times \prod_{i=1}^I \mathcal{U}_i$ in (2), and similarly in (1).

The theorem can be extended to an infinite number I of individuals, with the sums in the right-hand sides of (1) and (2) remaining finite. To this end it suffices to apply the current theorem to an artificial society made of a single individual whose preferences are endowed with the expected multi-utility representation $\mathcal{U} = \text{conv}\big(\bigcup_{i=1}^I \mathcal{U}_i\big)$, assuming $\text{cone}(\mathcal{U}) + \{\gamma e\}_{\gamma\in\mathbb{R}}$ is closed for part (ii). This provides a generalization of Zhou's (1997) aggregation theorem to incomplete preferences (in the case where $\mathcal{X}$ is finite).

Social preferences can be more complete than individual preferences and, in particular, $\succsim_0$ can be complete even though $\succsim_i$ is incomplete for all $i = 1, \ldots, I$. In this case, endowing $\succsim_0$ with an expected utility representation u_0, (1) reduces

[5] conv $(\cdot)$ denotes convex hull.

to $u_0 = \sum_{i=1}^{I} \alpha_i u_i - \beta_i v_i + \gamma e$ for some $(\alpha, \beta, \gamma, (u_i, v_i)_{i=1}^{I}) \in \mathbb{R}_+^{2I} \times \mathbb{R} \times \prod_{i=1}^{I} \mathcal{U}_i^2$, and (2) to $u_0 = \sum_{i=1}^{I} \theta_i u_i + \gamma e$ for some $(\theta, \gamma, (u_i)_{i=1}^{I}) \in \mathbb{R}_+^{I} \times \mathbb{R} \times \prod_{i=1}^{I} \mathcal{U}_i$. On the other hand, social preferences can also be less complete than individual preferences (in the extreme, the social preference relation can reduce to the Pareto-indifference or Pareto-preference relation) and, in particular, $\succsim_0$ can be incomplete even though $\succsim_i$ is complete for all $i = 1, \ldots, I$. In this case, endowing $\succsim_i$ with an expected utility representation u_i for all $i = 1, \ldots, I$, (1) reduces to $\mathcal{U}_0 = \left\{ \sum_{i=1}^{I} \theta_i u_i + \gamma e : (\theta, \gamma) \in \mathcal{W} \right\}$ for some convex set $\mathcal{W} \subseteq \mathbb{R}^I \times \mathbb{R}$, and (2) to the same with $\mathcal{W} \subseteq \mathbb{R}_+^I \times \mathbb{R}$.

These two particular cases (complete social preferences with incomplete individual preferences or the other way around) have in common that $\mathcal{L} = \mathcal{Y} \times \mathcal{G}$ for some convex sets $\mathcal{Y} \subseteq \mathbb{R}_+^{2I} \times \mathbb{R}$ and $\mathcal{G} \subseteq \prod_{i=1}^{I} \mathcal{U}_i^2$ in (1), and $\mathcal{M} = \mathcal{Z} \times \mathcal{H}$ for some convex sets $\mathcal{Z} \subseteq \mathbb{R}_+^I \times \mathbb{R}$ and $\mathcal{H} \subseteq \prod_{i=1}^{I} \mathcal{U}_i$ in (2). Such a separation between weights and utilities is not always possible. This can be shown from the example above if we now let $\mathcal{U}_0 = \operatorname{conv}\left(\left\{u_0^a, u_0^b\right\}\right)$, where $u_0^a = \frac{3}{4} u_1 + \frac{1}{4} u_2^a$ and $u_0^b = \frac{1}{4} u_1 + \frac{3}{4} u_2^b$. Then $(\succsim_i)_{i=0}^{I}$ clearly satisfies Pareto preference, yet any $\mathcal{M}$ satisfying (2) contains both $\left(\left(\frac{3}{4}, \frac{1}{4}\right), 0, \left(u_1, u_2^a\right)\right)$ and $\left(\left(\frac{1}{4}, \frac{3}{4}\right), 0, \left(u_1, u_2^b\right)\right)$ but neither $\left(\left(\frac{3}{4}, \frac{1}{4}\right), 0, \left(u_1, u_2^b\right)\right)$ nor $\left(\left(\frac{1}{4}, \frac{3}{4}\right), 0, \left(u_1, u_2^a\right)\right)$.

Seeking a general characterization, in terms of the preference profile $(\succsim_i)_{i=0}^{I}$, of the possibility of separating weights and utilities in the above sense does not seem a promising avenue of research. Such a separation can be obtained in a multi-profile setting, by means of an additional *independence of irrelevant alternatives* condition linking distinct profiles $(\mathcal{U}_i)_{i=0}^{I}$ with one another (Danan, Gajdos, and Tallon 2013). This latter principle, however, also implies that $\mathcal{G} = \prod_{i=1}^{I} \mathcal{U}_i^2$ in (1) and $\mathcal{H} = \prod_{i=1}^{I} \mathcal{U}_i$ in (2). It is an open problem to find weaker conditions allowing society to make a selection within the individual sets of utility functions (thereby reducing social incompleteness) while retaining the separation between weights and utilities.

Appendix

A. *On Expected Multi-Utility Representations*

The following lemma gathers useful properties of expected multi-utility representations. For a proof see Shapley and Baucells (1998, pp. 6–11) or Dubra, Maccheroni, and Ok (2004, pp. 128–131).

LEMMA 1: *A preference relation $\succsim$ over $\mathcal{P}$ admits an expected multi-utility representation if and only if there exists a closed and convex cone $\mathcal{K} \subseteq \mathbb{R}^{\mathcal{X}}$, $\mathcal{K} \perp \{\gamma e\}_{\gamma \in \mathbb{R}}$, such that for all $p, q \in \mathcal{P}$, $p \succsim q \Leftrightarrow p - q \in \mathcal{K}$. Moreover, $\mathcal{K}$ is unique, and a convex set $\mathcal{U} \subseteq \mathbb{R}^{\mathcal{X}}$ is an expected multi-utility representation of $\succsim$ if and only if* $\operatorname{cl}\left(\operatorname{cone}(\mathcal{U}) + \{\gamma e\}_{\gamma \in \mathbb{R}}\right) = \mathcal{K}^*$.[6]

[6] $\perp$ denotes orthogonality, $\operatorname{cl}(\cdot)$ denotes closure, and $\mathcal{K}^*$ denotes the dual cone of $\mathcal{K}$, i.e., $\mathcal{K}^* = \{u \in \mathbb{R}^{\mathcal{X}} : \forall k \in \mathcal{K}, \sum_{x \in \mathcal{X}} k(x) u(x) \geq 0\}$.

B. *Proof of the Theorem*

The "if" statements of both parts of the theorem are obvious. We only prove the "only if" statements.

We start with part (ii), so assume $\sum_{i=1}^{I} \text{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}}$ is closed and $(\succsim_i)_{i=0}^{I}$ satisfies Pareto preference. It is sufficient to show that for all $u_0 \in \mathcal{U}_0$, there exist $\theta \in \mathbb{R}_+^I$, $\gamma \in \mathbb{R}$, and $u_i \in \mathcal{U}_i$ for all $i = 1, \ldots, I$ such that $u_0 = \sum_{i=1}^{I} \theta_i u_i + \gamma e$. Indeed, if this claim is correct then the set

$$\mathcal{M} = \left\{(\theta, \gamma, (u_i)_{i=1}^{I}) \in \mathbb{R}_+^I \times \mathbb{R} \times \prod_{i=1}^{I} \mathcal{U}_i : \sum_{i=1}^{I} \theta_i \, u_i + \gamma e \in \mathcal{U}_0\right\}$$

satisfies (2) by construction and is θ—and $(u_i)_{i=1}^{I}$—sectionally convex since $\mathcal{U}_0$ is convex.

To prove the claim, let $\mathcal{K}_i$ be the closed and convex cone corresponding to $\succsim_i$ in Lemma 1, for all $i = 0, \ldots, I$. We then have $\cap_{i=1}^{I} \mathcal{K}_i \subseteq \mathcal{K}_0$ by Pareto preference and, hence, $\mathcal{K}_0^* \subseteq \left(\cap_{i=1}^{I} \mathcal{K}_i\right)^* = \text{cl}\left(\sum_{i=1}^{I} \mathcal{K}_i^*\right)$ (Rockafellar 1970, Corollary 16.4.2). Moreover, again by Lemma 1, $\mathcal{K}_i^* = \text{cl}\left(\text{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}}\right)$ for all $i = 0, \ldots, I$. Hence

$$\begin{aligned}
\mathcal{U}_0 &\subseteq \text{cl}\left(\text{cone}(\mathcal{U}_0) + \{\gamma e\}_{\gamma \in \mathbb{R}}\right) = \mathcal{K}_0^* \subseteq \text{cl}\left(\sum_{i=1}^{1} \mathcal{K}_1^*\right) \\
&\subseteq \text{cl}\left(\sum_{i=1}^{I} \text{cl}\left(\text{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}}\right)\right) \\
&= \text{cl}\left(\sum_{i=1}^{I} \left(\text{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}}\right)\right) \\
&= \text{cl}\left(\sum_{i=1}^{I} \text{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}}\right) \\
&= \sum_{i=1}^{I} \text{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}},
\end{aligned}$$

where the last equality follows from the assumption that $\sum_{i=1}^{I} \text{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}}$, is closed. Hence for all $u_0 \in \mathcal{U}_0$, there exist $\gamma \in \mathbb{R}$ and $u_i' \in \text{cone}(\mathcal{U}_i)$ for all $i = 1, \ldots, I$ such that $u_0 = \sum_{i=1}^{I} u_i' + \gamma e$. Moreover, for all $i = 1, \ldots, I$, since $\mathcal{U}_i$ is convex we also have $u'_i = \theta_i u_i$ for some $\theta_i \in \mathbb{R}_+$ and $u_i \in \mathcal{U}_i$ and, hence, $u_0 = \sum_{i=1}^{I} \theta_i u_i + \gamma e$.

Now for part (i), assume $(\succsim_i)_{i=0}^{I}$ satisfies Pareto indifference. As in part (ii) it is sufficient to show that for all $u_0 \in \mathcal{U}_0$, there exist $\alpha, \beta \in \mathbb{R}_+^I$, $\gamma \in \mathbb{R}$, and $u_i, v_i \in \mathcal{U}_i$ for all $i = 1, \ldots, I$ such that $u_0 = \sum_{i=1}^{I} \alpha_i u_i - \mu_i v_i + \gamma e$. To prove this, define the preference relation $\succsim_i'$ over $\mathcal{P}$ by $p \succsim_i' q \Leftrightarrow p \sim_i q$, for all $i = 1, \ldots, I$. We then have $p \succsim_i' q \Leftrightarrow p - q \in \mathcal{K}_i \cap (-\mathcal{K}_i)$, and $(\succsim_0, (\succsim_i')_{i=1}^{I})$ obviously satisfies Pareto preference, so by the same argument as in the proof of part (ii) we obtain $\mathcal{K}_0^* \subseteq \mathrm{cl}\big(\sum_{i=1}^{I}(\mathcal{K}_i \cap (-\mathcal{K}_i))^*\big) = \mathrm{cl}\big(\sum_{i=1}^{I} \mathrm{cl}\,(\mathcal{K}_i^* - \mathcal{K}_i^*)\big)$ $= \mathrm{cl}\big(\sum_{i=1}^{I}(\mathcal{K}_i^* - \mathcal{K}_i^*)\big)$ (Rockafellar 1970, Corollary 16.4.2). Hence

$$\begin{aligned}
\mathcal{U}_0 &\subseteq \mathrm{cl}\big(\mathrm{cone}(\mathcal{U}_0) + \{\gamma e\}_{\gamma\in\mathbb{R}}\big) = \mathcal{K}_0^* = \mathrm{cl}\Big(\sum_{i=1}^{1}(\mathcal{K}_1^* - \mathcal{K}_i^*)\Big) \\
&\subseteq \mathrm{cl}\Big(\sum_{i=1}^{I}\big(\mathrm{cl}\big(\mathrm{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma\in\mathbb{R}}\big) - \mathrm{cl}\big(\mathrm{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma\in\mathbb{R}}\big)\big)\Big) \\
&= \mathrm{cl}\Big(\sum_{i=1}^{I}\big(\mathrm{cone}(\mathcal{U}_i) - \mathrm{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma\in\mathbb{R}}\big)\Big) \\
&= \mathrm{cl}\Big(\sum_{i=1}^{I}(\mathrm{cone}(\mathcal{U}_i) - \mathrm{cone}(\mathcal{U}_i)) + \{\gamma e\}_{\gamma\in\mathbb{R}}\Big) \\
&= \sum_{i=1}^{I}(\mathrm{cone}(\mathcal{U}_i) - \mathrm{cone}(\mathcal{U}_i)) + \{\gamma e\}_{\gamma\in\mathbb{R}} \\
&= \sum_{i=1}^{I}\mathrm{cone}(\mathcal{U}_i) - \sum_{i=1}^{I}\mathrm{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma\in\mathbb{R}},
\end{aligned}$$

where the before-last equality follows from the fact that $\mathrm{cone}(\mathcal{U}_i) - \mathrm{cone}(\mathcal{U}_i)$ and $\{\gamma e\}_{\gamma\in\mathbb{R}}$ are subspaces of $\mathbb{R}^{\mathcal{X}}$. Hence for all $u_0 \in \mathcal{U}_0$, there exist $\gamma \in \mathbb{R}$ and $u_i', v_i' \in \mathrm{cone}(\mathcal{U}_i)$ for all $i = 1, \ldots, I$ such that $u_0 = \sum_{i=1}^{I} u_i' - v_i' + \gamma e$. Moreover, for all $i = 1, \ldots, I$, since $\mathcal{U}_i$ is convex we also have $u_i' = \alpha_i u_i$ and $v_i' = \beta_i v_i$ for some $\alpha_i, \beta_i \in \mathbb{R}_+$ and $u_i, v_i \in \mathcal{U}_i$ and, hence, $u_0 = \sum_{i=1}^{I} \alpha_i u_i - \beta_i v_i + \gamma e$. ∎

C. *On the Closedness Assumption in Part of the Theorem*

As can be seen from the proof of part (ii), the closedness assumption ensures that each social utility function can be expressed as a nonnegative linear combination of some individual utility functions (plus a constant function). Without this assumption, each social utility function can only be expressed as the limit of a sequence of such combinations.

For an example in which the assumption is not satisfied and (2) does not hold for any $\mathcal{M}$, let $\mathcal{X} = \{x, y, z, w\}$, $I = 2$, $\mathcal{U}_0 = \{u_0\}$, $\mathcal{U}_1 = \{u_1\}$, and $\mathcal{U}_2 = \{u_2(s,t)\colon\ s, t \in \mathbb{R}, s^2 + t^2 \le 1\}$, where $u_0, u_1, u_2(s,t)$ are as follows:

	u_0	u_1	$u_2(s,t)$
x	1	-1	1
y	-1	1	s
z	1	0	t
w	-1	0	$-1 - s - t$.

Then $\sum_{i=1}^{2} \mathrm{cone}(\mathcal{U}_i) = \{u \in \mathbb{R}^{\mathcal{X}} : u(x) + u(y) \geq 0,\ u(z) = 0 \text{ or } u(x) + u(y) > 0,\ u(x) + u(y) + u(z) + u(w) = 0\}$ and, hence, $\sum_{i=1}^{2} \mathrm{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}} = \{u \in \mathbb{R}^{\mathcal{X}} : u(x) + u(y) \geq u(z) + u(w),\ \ 3u(z) = u(x) + u(y) + u(w) \text{ or } u(x) + u(y) > u(z) + u(w)\}$. This latter set is not closed, and indeed u_0 does not belong to it but belongs to its closure. Hence u_0 cannot be expressed as a nonnegative linear combination of u_1 and some $u_2(s,t) \in \mathcal{U}_2$ even though $(\succsim_i)_{i=0}^{2}$ satisfies Pareto preference. The same conclusion would be reached with any other expected multi-utility representation of $\succsim_i$ for all $i = 0, 1, 2$.

A sufficient condition for the existence of a profile $(\mathcal{U}_i)_{i=1}^{I}$ satisfying the closedness assumption is that $\sum_{i=1}^{I} \mathcal{K}_i^*$ be closed, where $\mathcal{K}_i$ is the closed and convex cone corresponding to $\succsim_i$ in Lemma 1 (one can then take $\mathcal{U}_i = \mathcal{K}_i^*$, for instance). There are at least two cases where this sufficient condition is always satisfied.

The first case is when each $\mathcal{K}_i$ is polyhedral (Rockafellar 1970, Corollary 19.2.2, 19.3.2). This can be characterized by a *finiteness* axiom on $\succsim_i$ (Dubra and Ok 2002).[7] Note that no closedness assumption is needed in part (i) because $\mathrm{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}}$ is replaced with $\mathrm{cone}(\mathcal{U}_i) - \mathrm{cone}(\mathcal{U}_i) + \{\gamma e\}_{\gamma \in \mathbb{R}}$, which is a subspace of $\mathbb{R}^{\mathcal{X}}$ and, hence, falls into this case.

The second case is when all $\mathcal{K}_i$'s have a common point in their relative interiors (Rockafellar 1970, Corollary 16.4.2). This can be characterized by the following *minimal agreement* condition: there exist $p, q \in \mathcal{P}$ such that $p \succsim_i^* q$ for all $i = 1, \ldots, I$, where $p \succsim_i^* q$ is defined by for all $q_i \in \mathcal{P}$ such that $p \succsim_i q_i$, there exist $q_i' \in \mathcal{P}$ and $\lambda_i \in (0,1)$ such that $p \succsim_i q_i'$ and $q = \lambda_i q_i + (1 - \lambda_i) q_i'$. Note that if all $\succsim_i$s are complete then this condition boils down to the usual minimal agreement condition, where $p \succsim_i^* q$ is replaced with $p \succ_i q$.

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[7] In this case an alternative proof of the theorem consists in considering each extreme point of each individual's expected multi-utility representation as an expected utility representation of an artificial individual with complete preferences and applying Harsanyi's aggregation theorem to the artificial society.

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