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Social rationality, separability, and equity under uncertainty

Marc Fleurbaey [a], Thibault Gajdos [b], Stéphane Zuber [c,*]

[a] Woodrow Wilson School and Center for Human Values, Princeton University, Wallace Hall, Princeton University, Princeton, NJ 08544, USA

[b] Greqam, CNRS, EHESS, Université d'Aix-Marseille (Aix Marseille School of Economics), Centre de la Vieille Charité, 2 rue de la Charité, 13236 Marseille Cedex 02, France

[c] Paris School of Economics – CNRS, Centre d'Economie de la Sorbonne, Maison des Sciences Economiques, 106-112 Boulevard de l'Hôpital, 75647 Paris Cedex 13, France

HIGHLIGHTS

- Following Harsanyi (1955), we study the aggregation of preferences under risk.
- We allow ex post welfare to depend on ex ante prospects and counterfactuals.
- Ex ante and ex post equity are incorporated, contrasting with Harsanyi's approach.
- We highlight the remaining difficulty to obtain separable aggregations of preferences.
- For weak notions of separability, we however find a rich configuration of criteria.

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ABSTRACT

Harsanyi (1955) proved that, in the context of uncertainty, social rationality and the Pareto principle impose severe constraints on the degree of priority for the worst-off that can be adopted in the social evaluation. Since then, the literature has hesitated between an ex ante approach that relaxes rationality (Diamond, 1967) and an ex post approach that fails the Pareto principle (Hammond, 1983; Broome, 1991). The Hammond–Broome ex post approach conveniently retains the separable form of utilitarianism but does not make it explicit how to give priority to the worst-off, and how much disrespect of individual preferences this implies. Fleurbaey (2010) studies how to incorporate a priority for the worst-off in an explicit formulation, but leaves aside the issue of ex ante equity in lotteries, retaining a restrictive form of consequentialism. We extend the analysis to a framework allowing for ex ante equity considerations to play a role in the ex post evaluation, and find a richer configuration of possible criteria. But the general outlook of the Harsanyian dilemma is confirmed in this more general setting.

1. Introduction

Harsanyi (1955) published a theorem that has attracted a lot of attention. The theorem says that in the context of lotteries, if the individuals and the social observer are expected utility maximizers, the Pareto principle applied to individual expected utilities over lotteries implies that the von Neumann–Morgenstern (VNM) utility of the social observer is affine with respect to the vector of individual VNM utilities. While Harsanyi viewed this result as a key argument in a justification of utilitarianism, many commentators understood it as a negative result. Namely, the combination of respect for individual preferences (Pareto) and social rationality (expected utility on behalf of the social observer) imposes severe constraints on the degree of priority for the worst-off. The social observers who want to be more egalitarian than allowed by Harsanyi's theorem have to choose between irrationality and paternalism, two great evils in mainstream welfare economics.

Indeed, several papers in the literature following Harsanyi's paper have showed the impossibility to satisfy the Pareto principle *ex ante*, which is a principle of non paternalism requiring individual choices to be respected, and a weak principle of social rationality, namely dominance (recent references generalizing these findings include Gajdos et al., 2008; Fleurbaey, 2009; Fudenberg and Levine, 2012; Chambers and Hayashi, 2014). This is true even in non expected utility frameworks, provided we stick to consequentialism, that is if we assume that welfare in each state of

* Corresponding author. Tel.: +33 1 44078917.
E-mail addresses: mfleurba@princeton.edu (M. Fleurbaey), thibault.gajdos@univ-amu.fr (T. Gajdos), stephane.zuber@univ-paris1.fr (S. Zuber).

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the world only depends on consequences in this state of the world, and not on counterfactual events (consequences that could have happened and the risk that was borne *ex ante*). Given this tradeoff, the literature has often hesitated between an *ex ante* approach in which an inequality-averse social criterion is applied to individual expected utilities, and an *ex post* approach in which one computes the expected value of an inequality-averse social welfare function. The former approach violates social rationality, as noted by Machina (1989) and Grant (1995), in particular it violates dynamic consistency and stochastic dominance. While a fix can be proposed for dynamic consistency (Epstein and Segal, 1992), the issue of dominance is more serious. The latter approach violates Pareto *ex ante*, and is often criticized for being unable to account for *ex ante* fairness through a randomization mechanism (Diamond, 1967).

One solution to the dilemma is to use a combination of an *ex ante* and *ex post* criterion. This route was followed in the literature on the measurement of inequality under uncertainty (Ben Porath et al., 1997; Gajdos and Maurin, 2004; Chew and Sagi, 2012).[1] This branch of the literature has proposed interesting criteria, but they typically combine the problems of the *ex ante* and *ex post* approaches, by violating both the Pareto principle and dominance, and by being non-separable in all respects: in states of the world and individuals. They also directly focus on the distribution of income or wealth and do not explicitly deal with individuals' welfare measures. In the present paper, we want to be more explicit in that respect, by assuming that individuals' welfare in each state of the world may depend on counterfactuals, and therefore dropping the consequentialist assumption of most of the literature.

Hammond (1983) and Broome (1991) have indeed proposed a version of the ex post approach in which the form of Harsanyi's utilitarianism is retained, but individual utilities are reinterpreted as the contribution to social welfare brought by individual situations. This approach is most elegant, but these authors have not explored in detail how to incorporate a priority for the worst-off in the measurement of individual utilities, and how much divorce with individual preferences over lotteries this would imply. As a consequence, their theories remain too abstract and implicit for concrete applications. A natural step, in this respect, is to adopt a richer description of the consequences, so that one can make a difference between final consequences obtained with or without a fair lottery. With such a richer description of consequences, Diamond's critique seems powerless against the ex post approach which is then able to combine rationality and a concern for ex ante fairness.[2]

Hammond (1981) also observed that when individual beliefs on probabilities are not trustworthy, respecting their ex ante preferences is not as compelling as in the case of full information. Fleurbaey (2010) argued that this is actually the general case, as probabilistic beliefs are generally different from actual probabilities. Truly enough, the social observer's own beliefs may not be much more reliable in general. However, Fleurbaey noted that in situations of randomized prizes as in Diamond's example, there is an interesting difference between a social observer who is sure of the final distribution of utilities and individuals who do not know their own final utility. Such situations are not risky for the observer and this may justify disrespecting individual preferences: preventing individuals from taking some risk is in the interest of the future losers, who are bound to exist and are ex ante ignorant of their true interests. This line of argument may justify weakening the application of the Pareto principle *ex ante*.

Dropping the Pareto principle, however, does not fully eliminate the difficulty. The argument of the previous paragraph only applies in cases of sure inequalities and the Pareto principle remains compelling when equality is preserved in all possible consequences. Fleurbaey shows that retaining the Pareto principle in cases of perfect equality and combining it with dominance singles out a social criterion: maximizing the expected value of the equally-distributed equivalent utility (Atkinson, 1970). This criterion is nice in several respects but it is strongly non-separable across subpopulations, as the equally-distributed equivalent utility in a state of nature will typically depend on the whole vector of utilities in that state. Therefore, the bulk of Harsanyi's theorem is preserved if one adds a requirement of separability across subpopulations.

Some separability across subpopulations may seem desirable in practical applications. Indeed, if one considers a dynamic framework, one may want to be able to make decisions for future risks independently of the utility of those who have lived in the distant past. One reason is informational parsimony and the difficulty to know the distribution of past utilities. The requirement of "Independence of the Utility of the Dead", introduced in Blackorby et al. (2005), thus seems attractive if only for practical convenience. In summary, the dilemma for an inequality-averse social observer seems to involve three evils rather than just two: irrationality, paternalism, non-separability. Another possible argument for separability is that one may hold that all affected people, and only them, should have a say in decisions where their interests are at stake. Hence if only a subgroup of people is affected in a choice between two courses of action, it may seem appealing to only consider the utilities of people in this subgroup when making the collective decision.

Fleurbaey's analysis shares with the Hammond–Broome theory the unpalatable feature that it is not fully explicit. While it is explicit about inequality aversion, it leaves it implicit how to incorporate a concern for ex ante fairness in the measurement of final utilities. Formally, it retains a narrow form of consequentialism in which the evaluation of ex post consequences in a particular state of nature only involves the utilities obtained in this state of nature. The interplay between ex ante fairness and ex post inequality aversion is therefore left unexplored. In this paper, we set out to analyze the form of the dilemma when the evaluation of ex post consequences may involve the counterfactual utilities of other states of nature. Formally, this means that the requirement of dominance becomes much less constraining.

We also extend the analysis in another direction. Unlike many papers pursuing Harsanyi's work,[3] we will not assume that the evaluation of ex ante individual prospects, as referred to in the Pareto principle, is based on expected utility. In this way the analysis gains in generality and the negative results, if any, become even more problematic. Our results are not totally negative but they show that the essence of the dilemma remains. More precisely, we show that the combination of rationality and separability imposes such constraints on the social criterion that the dilemma between paternalism and priority for the worst-off is unescapable.

The structure of the paper is straightforward. In Section 2 the framework is presented, followed in Section 3 by the axioms that embody the requirements we want to impose on the social criterion. The results are stated in Section 4 and discussed in Section 5. Section 6 concludes. All the proofs are gathered in the Appendix.

1 A similar route was followed recently by Saito (2013) in the context of other individual preferences for fairness.

2 See Adler and Sanchirico (2006) for a rich discussion of these issues and an endorsement of the ex post approach.

3 Notable exceptions are Blackorby et al. (2004) and Gajdos et al. (2008).

2. Setup

The framework involves state-contingent alternatives,[4] with a finite set of states of nature $\mathcal{S} = \{1, \ldots, s\}$. The population is a finite set $\mathcal{N} = \{1, \ldots, n\}$.

The objects of evaluation are prospects (u, z), in which $u \in \mathcal{U} = \mathbb{R}^{ns}$ is a utility matrix such that $u_{i\sigma}$ is the utility obtained by $i \in \mathcal{N}$ in state $\sigma \in \mathcal{S}$, and $z \in \mathcal{Z}$ denotes a relevant ex ante non-utility information about states of nature, namely an information on the probabilities of the states of nature. To be more specific, letting $\mathcal{P} = \left\{(p_1, \ldots, p_s) \in (0, 1)^s : \sum_{\sigma\in\mathcal{S}} p_\sigma = 1\right\}$, i.e., the $(s-1)$-simplex, we assume that $\mathcal{Z}$ is the set of non-empty subsets of $\mathcal{P}$, $\mathcal{Z} = 2^{\mathcal{P}}$. Hence $z \in \mathcal{Z}$ is a set of possible probabilities of occurrence of the different states of nature (null probabilities are excluded). Let $z^* = \left\{\left(\frac{1}{s}, \ldots, \frac{1}{s}\right)\right\}$, meaning that the observer is sure that the states of nature are equiprobable.

Let $u_i = (u_{i\sigma})_{\sigma\in\mathcal{S}}$ and $u_\sigma = (u_{i\sigma})_{i\in\mathcal{N}}$. Let $u_{-i} = (u_j)_{j\in\mathcal{N}\setminus\{i\}}$, and for $M \subseteq \mathcal{N}$, $u_M = (u_i)_{i\in M}$. The subset of sure prospects, i.e., of prospects u such that $u_\sigma = u_\tau$ for all $\sigma, \tau \in \mathcal{S}$, is denoted $\mathcal{U}^c$. The subset of egalitarian prospects, i.e., of prospects u such that $u_i = u_j$ for all $i, j \in \mathcal{N}$, is denoted $\mathcal{U}^e$.

The utility figure $u_{i\sigma}$ must be interpreted as measuring the utility obtained by individual i in state σ, without consideration of inequality in society or fairness in the lottery. The goal of this paper is to define how to incorporate such considerations explicitly in the social evaluation. In contrast, the value of $u_{i\sigma}$ may include everything that is relevant in i's personal ex post situation, including the utility consequences of bearing risk in the ex ante situation. For instance, if i has taken a great risk and suffered anxiety, this may yield a low $u_{i\sigma}$ even in a lucky state of nature. We do not explicitly model individual preferences under uncertainty and the underlying economic allocations. We work directly with utility consequences.

Ex ante, the social planner faces a prospect $(u, z) \in \mathcal{U} \times \mathcal{Z}$. Ex post, the social planner faces a situation $(u, z, \sigma) \in \mathcal{U} \times \mathcal{Z} \times \mathcal{S}$. We are interested in three preference orderings, which are all supposed to belong to the same ethical observer who seeks to make a coherent assessment of ex ante prospects and ex post consequences.

- Ex post preferences on individual situations, denoted R, over $\mathbb{R}^s \times \mathcal{Z} \times \mathcal{S}$.

  Such preferences do not consider only the ex post utility $u_{i\sigma}$ in the current state of the world σ, but the whole vector (u_i, z, σ) containing counterfactual information about the utility in other states and the probability of the different states of the world. This is so because what could have happened in other (non-realized) states of nature may be important in order to assess whether the individual has been fairly treated.
- Ex post preferences on social situations, denoted R^p, over $\mathcal{U} \times \mathcal{Z} \times \mathcal{S}$.

  Again, such preferences do not only consider the ex post utility vector in state σ, u_σ, because utility in counterfactual states may carry relevant information.
- Ex ante preferences on social prospects, denoted R^a, over $\mathcal{U} \times \mathcal{Z}$.

Let P and I denote the strict preference and indifference relations, respectively, corresponding to R. The relations P^p, I^p, P^a, and I^a are defined similarly.

Two clarificatory remarks must be made here. First, we need three orderings, not just one, even though the objective is ultimately to find a good ex ante ordering R^a. The role of R and R^p is essential in the explicit formulation of individualism (which relates R^p to R) and of rationality (which, in the principle of dominance, relates R^a to R^p).

Second, the ex post orderings R, R^p do not just rank final consequences conditionally on a given state σ and given probabilities. They also compare final consequences pertaining to different states and probabilities. For this purpose, one must interpret $(u, z, \sigma)\, R^p \left(u', z', \sigma'\right)$ as telling something about the joint contribution of utility and probability to the value of the final consequence.

3. Axioms

Our framework is extremely general and accommodates all sorts of possible preferences, many of which are unpalatable. We therefore need quite a few axioms in order to exclude such unacceptable preferences at the ex post or at the ex ante stage.

The axioms we want to impose on this triple of orderings fall under three headings: social rationality, individualism and separability. We do not introduce specific axioms that would capture the ideals of priority for the worst-off and ex ante fairness. The results we obtain make it clear how such ideals can be satisfied in combination with the axioms studied in this paper. This will be discussed in Section 5.

3.1. Social rationality

Harsanyi (1955) requires the social criterion to take the form of expected welfare. We also make rationality assumptions, but in a more general form that encompasses non-expected utility criteria and turns out, as we shall see, to be formally weak enough to accommodate the ex ante approach. First, the relations under consideration should be complete and continuous pre-orders.

Axiom 1 (*Ordering*)**.** The three relations R, R^p, R^a are transitive, reflexive, complete, and continuous.[5]

The key rationality axioms are dominance and independence. Dominance means that an improvement in all possible consequences for the different states of nature must yield a global improvement—recall that there are no null states. This is really the minimal requirement of social rationality.[6]

Axiom 2 (*Dominance*)**.** For all $u, u' \in \mathcal{U}$, $z, z' \in \mathcal{Z}$,

$$\left[\forall\sigma \in \mathcal{S},\ (u, z, \sigma)R^p(u', z', \sigma)\right] \Rightarrow (u, z)R^a(u', z'),$$

and

$$\left.\begin{array}{ll}\forall\sigma \in \mathcal{S}, & (u, z, \sigma)R^p(u', z', \sigma)\\ \exists\hat{\sigma} \in \mathcal{S}, & (u, z, \hat{\sigma})P^p(u', z', \hat{\sigma})\end{array}\right\} \Rightarrow (u, z)P^a(u', z').$$

Independence is formulated here in a way that remains compatible with many non-expected utility approaches, because it is applied in a way that takes account of the whole matrix u in the evaluation of ex post consequences. This is therefore a rather weak axiom.

Axiom 3 (*Independence*)**.** For all $u, v, u', v' \in \mathcal{U}$, $y, z, y', z' \in \mathcal{Z}$ and all $T \subseteq \mathcal{S}$,

$$\left.\begin{array}{ll}\forall\sigma \in T, & (u, y, \sigma)I^p(v, z, \sigma)\\ \forall\sigma \in T, & (u', y', \sigma)I^p(v', z', \sigma)\\ \forall\sigma \in \mathcal{S}\setminus T, & (u, y, \sigma)I^p(u', y', \sigma)\\ \forall\sigma \in \mathcal{S}\setminus T, & (v, z, \sigma)I^p(v', z', \sigma)\end{array}\right\} \Rightarrow \left[\begin{array}{c}(u, y)R^a(v, z)\\ \Leftrightarrow\\ (u', y')R^a(v', z').\end{array}\right].$$

4 For the adaptation of Harsanyi's theorem to such a framework, see Blackorby et al. (1999).

5 A relation $\tilde{R}$ on X is said to be continuous if, for all $x \in X$, the sets $\{y \in X \mid x\tilde{R}y\}$ and $\{y \in X \mid y\tilde{R}x\}$ are closed.

6 Observe that this axiom applies to prospects with different probabilities, which is sensible if one recalls that $(u, z, \sigma)R^p(u', z', \sigma)$ incorporates the contribution of probabilities to the value of the final consequence.

The next axiom is meant to rule out degenerate criteria for which the evaluation of ex post consequences is only based on ex ante information. In particular, we want to avoid a case where $(u, z, \sigma)I^p(u, z, \sigma')$ for all $u \in \mathcal{U}$ and all $z \in \mathcal{Z}$. Introduced by Skiadas (1997) under the label 'Solvability', the axiom requires sufficient richness in the possible evaluation of the ex post consequences of a given prospect.[7]

Axiom 4 (*Ex Post Richness*)**.** For all $z \in \mathcal{Z}$ and for any collection of s utility matrices $u^1 \in \mathcal{U}, \ldots, u^s \in \mathcal{U}$, there exists $u \in \mathcal{U}$ such that for all $\sigma \in \mathcal{S}$ and $i \in \mathcal{N}$, $(u_i, z, \sigma)I(u_i^\sigma, z, \sigma)$.

We also introduce axioms that require a natural degree of simplicity in the evaluation of final situations in different states. First, we require the role of states to be symmetric in the ex ante evaluation, as observed for instance in expected utilities which are sums of terms representing the contribution of each state to the expected value, each term being the product of the probability of the state by the utility attained in the state.[8]

Axiom 5 (*State Neutrality*)**.** For all $u, u' \in \mathcal{U}, z, z' \in \mathcal{Z}$, if there exists a permutation $\pi : \mathcal{S} \to \mathcal{S}$ such that $(u, z, \sigma)I^p(u', z', \pi(\sigma))$ for all $\sigma \in \mathcal{S}$, then $(u, z)I^a(u', z')$.

The next axiom rules out ex post preferences that would embody a pure preference for ending up in a particular state. Pure preferences for states are not easily excluded in our framework, because the object (u_i, z, σ) indicates that one is in state σ only through the third term, while u_i does not make it explicit. This marks a difference with the narrow consequentialist setting in which the ex post situation is described by $u_{i\sigma}$, so that it is easy to impose that ex post preferences depend only on utility $u_{i\sigma}$ and not on σ. To exclude the possibility of a pure preference for states of nature, we require that for riskless prospects, the contribution of each state to ex ante evaluation is equal for a specific informational configuration, namely, equiprobable states of nature.

Axiom 6 (*State Equivalence*)**.** For all $u \in \mathcal{U}^c, \sigma, \sigma' \in \mathcal{S}, i \in \mathcal{N}$

$$(u_i, z^*, \sigma)I(u_i, z^*, \sigma').$$

3.2. Individualism

In Harsanyi (1955), individualism is embodied in the Pareto principle applied to ex ante prospects. As noted in Hammond (1981) and emphasized in Fleurbaey (2010), the Pareto principle is not compelling when applied to uncertain prospects because unanimity among future winners and losers may be obtained only because they ignore their ultimate interests. We therefore limit the application of this principle to ex post consequences, in which full information prevails.

Axiom 7 (*Ex Post Pareto*)**.** For all $u, u' \in \mathcal{U}, z, z' \in \mathcal{Z}$ and $\sigma, \sigma' \in \mathcal{S}$,

$$\left[\forall i \in \mathcal{N},\ (u_i, z, \sigma)R(u_i', z', \sigma')\right] \Rightarrow (u, z, \sigma)R^p(u', z', \sigma'),$$

and

$$\left.\begin{array}{ll}\forall i \in \mathcal{N}, & (u_i, z, \sigma)R(u_i', z', \sigma') \\ \exists i \in \mathcal{N}, & (u_i, z, \sigma)P(u_i', z', \sigma')\end{array}\right\} \Rightarrow (u, z, \sigma)P^p(u', z', \sigma').$$

We also introduce a monotonicity axiom made of two parts. The first one is standard, and requires that the evaluation of individual situations is increasing in the components of the prospects.[9] The second part of the axiom essentially requires that differences in information can always be compensated in the evaluation of individual situations by increasing or decreasing the prospect itself. Formally, this axiom is stated as follows.[10]

Axiom 8 (*Monotonicity*)**.**

1. For all $u, u' \in \mathcal{U}, z, z' \in \mathcal{Z},\ \sigma \in \mathcal{S}, i \in \mathcal{N}$,
   $u_i > u_i' \Rightarrow (u_i, z, \sigma)P(u_i', z, \sigma).$
2. For all $u_i \in \mathbb{R}^s, \sigma \in \mathcal{S}$ and $z \in \mathcal{Z}$, there exist two sure vectors $\underline{u}_i$ and $\bar{u}_i$ for which
   $(\bar{u}_i, z^*, \sigma)R(u_i, z, \sigma)R(\underline{u}_i, z^*, \sigma).$

Finally, we require individuals to be treated equally.

Axiom 9 (*Anonymity*)**.** For all $u, u' \in \mathcal{U}, z \in \mathcal{Z}, \sigma \in \mathcal{S}$, if there exists a permutation $\pi : \mathcal{N} \to \mathcal{N}$ such that for all $i \in \mathcal{N}, u_i' = u_{\pi(i)}$,

$$(u, z, \sigma)I^p(u', z, \sigma).$$

3.3. Separability

Harsanyi (1955) derives a separable (indeed, additive) social ordering from the combination of social rationality and ex ante Pareto. With the axioms introduced so far very little separability is obtained, and it appears interesting to study a quite attractive principle of separability. This principle says that individuals who are not affected and bear no risk should not influence the social evaluation. Individuals are not affected when their personal situation is the same in the prospects under consideration (but their prospects may still be risky). Individuals bear no risk when their personal situation is the same in all states of the world for each prospect under consideration (but the situation may be different for different prospects).

This principle is inspired by the observation that in a dynamic framework, if separability is not satisfied, one should either take account of the utility of dead people in the evaluation of prospects,[11] or ignore it and violate dynamic consistency. Our framework is not explicitly a dynamic one, but clearly our axiomatics would apply to the case of individuals belonging to successive generations.

We introduce two axioms capturing this idea. The first literally embodies the separability principle as just stated.

Axiom 10 (*Independence of the Utility of the Sure*)**.** For all $u, u' \in \mathcal{U}, v, v' \in \mathcal{U}^c, z, z' \in \mathcal{Z}, M \subset \mathcal{N}$,

$$((u_M, v_{\mathcal{N}\setminus M}), z)R^a((u_M', v_{\mathcal{N}\setminus M}), z')$$
$$\Longleftrightarrow$$
$$((u_M, v'_{\mathcal{N}\setminus M}), z)R^a((u_M', v'_{\mathcal{N}\setminus M}), z').$$

The second says that when the subgroup that takes risks and is affected is perfectly egalitarian in all possible states of nature, then the evaluation should proceed as if the whole society was doing the same. This means that, in this special case, the mere presence of unaffected and risk-free individuals has no influence on the evaluation, a property that is not guaranteed under the previous axiom.

Axiom 11 (*Restricted Independence of the Sure*)**.** For all $u, u' \in \mathcal{U}^e, v \in \mathcal{U}^c, z, z' \in \mathcal{Z}, M \subset \mathcal{N}, M \neq \varnothing$,

$$((u_M, v_{\mathcal{N}\setminus M}), z)R^a((u_M', v_{\mathcal{N}\setminus M}), z') \Leftrightarrow (u, z)R^a(u', z').$$

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[7] We have preferred the label 'Ex post richness' which in our view better expresses why the axiom is socially desirable. The label 'Solvability' seems exclusively technical and does not convey the idea that ex post evaluation is not the mere expression of an ex ante judgment.

[8] The expression $(u, z, \sigma)I^p(u', z', \pi(\sigma))$ incorporates the contribution of probabilities: it may be that in $\pi(\sigma)$, (u', z') has a greater utility but a lower probability than (u, z) in σ.

[9] Recall that by assumption, for all z in $\mathcal{Z}$, there is no null state.

[10] For two vectors x, y, $x > y$ means that $x \geq y$ and $x \neq y$.

[11] The principle of "independence of the utility of the dead" has been introduced by Blackorby et al. (2005). It is also invoked in Bommier and Zuber (2008).

4. Two families of social criteria

In the standard consequentialist framework, Axiom 10 (Independence of the Utility of the Sure) implies the following strong form of ex post separability (see Blackorby et al., 2005; Bommier and Zuber, 2008), which can also be justified normatively (see Broome, 1991).

Axiom 12 (*Ex Post Separability*). For all $u, v, u', v' \in \mathcal{U}, z, z' \in \mathcal{Z}$, $s, s' \in \mathcal{S}$, and $M \subset \mathcal{N}$,

$$\left.\begin{array}{ll} \forall i \in M, & u_i = v_i \\ \forall i \in M, & u'_i = v'_i \\ \forall i \in \mathcal{N} \setminus M, & (u_i, z, \sigma) I (u'_i, z', \sigma') \\ \forall i \in \mathcal{N} \setminus M, & (v_i, z, \sigma) I (v'_i, z', \sigma') \end{array}\right\} \Rightarrow \left[\begin{array}{c} (u, z, \sigma) R^p \left(u', z', \sigma'\right) \\ \Leftrightarrow \\ (v, z, \sigma) R^p (v', z', \sigma') \end{array}\right].$$

A similar implication can be obtained in our extended framework, as shown by the following lemma.

Lemma 1. *Axioms 1 (Ordering), 2 (Dominance), 6 (State Equivalence), 7 (Ex Post Pareto), 8 (Monotonicity) and 10 (Independence of the Utility of the Sure) imply Axiom 12 (Ex Post Separability).*

We are now ready to state our first main result. The principles introduced in Section 3 make it possible to single out two broad families of social criteria.

Proposition 1. *If Axioms 1 (Ordering), 2 (Dominance), 3 (Independence), 4 (Ex Post Richness), 5 (State Neutrality), 6 (State Equivalence), 7 (Ex Post Pareto), 8 (Monotonicity) and 10 (Independence of the Utility of the Sure) are satisfied, then:*

1. *there exists a continuous function $\varphi : \mathbb{R}^s \times \mathcal{Z} \times \mathcal{S} \to \mathbb{R}$, increasing in its first s arguments and satisfying $\varphi(u_i, z^*, \sigma) = \varphi(u_i, z^*, \sigma')$ for all $\sigma, \sigma' \in \mathcal{S}$ and $u \in \mathcal{U}^c$, such that for all $u_i, u'_i \in \mathbb{R}^s$, $z, z' \in \mathcal{Z}, \sigma, \sigma' \in \mathcal{S}$,*
$$(u_i, z, \sigma) R \left(u'_i, z', \sigma'\right) \Leftrightarrow \varphi(u_i, z, \sigma) \geq \varphi\left(u'_i, z', \sigma'\right);$$
2. *there exist n continuous increasing functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ such that for all $u, u' \in \mathcal{U}, z, z' \in \mathcal{Z}, \sigma, \sigma' \in \mathcal{S}$,*
$$(u, z, \sigma) R^p (u', z', \sigma') \Leftrightarrow \sum_{i \in \mathcal{N}} \varphi_i \circ \varphi(u_i, z, \sigma) \geq \sum_{i \in \mathcal{N}} \varphi_i \circ \varphi(u'_i, z', \sigma');$$
3. *one of* (a) *or* (b) *holds:*
   (a) *For all $u, u' \in \mathcal{U}$, $z, z' \in \mathcal{Z}$,*
$$(u, z) R^a (u', z') \Leftrightarrow \sum_{\sigma \in \mathcal{S}} \sum_{i \in \mathcal{N}} \varphi_i \circ \varphi(u_i, z, \sigma) \geq \sum_{\sigma \in \mathcal{S}} \sum_{i \in \mathcal{N}} \varphi_i \circ \varphi(u'_i, z', \sigma')$$
   and for all $i \in \mathcal{N}$ there exist a continuous increasing function ψ_i and a continuous function ξ_i such that for all $u \in \mathcal{U}^c$, $\sum_{\sigma \in \mathcal{S}} \varphi_i \circ \varphi(u_i, z, \sigma) = \psi_i(u_{i1}) + \xi_i(z)$.
   (b) *There exists $\alpha \neq 0$ such that for all $u, u' \in \mathcal{U}$, $z, z' \in \mathcal{Z}$,*
$$(u, z) R^a (u', z') \Leftrightarrow \sum_{\sigma \in \mathcal{S}} \alpha \exp\left(\alpha \sum_{i \in \mathcal{N}} \varphi_i \circ \varphi(u_i, z, \sigma)\right) \geq \sum_{\sigma \in \mathcal{S}} \alpha \exp\left(\alpha \sum_{i \in \mathcal{N}} \varphi_i \circ \varphi(u'_i, z', \sigma')\right)$$
   and for all $i \in \mathcal{N}$ there exist a continuous increasing function ϑ_i and a continuous function η_i such that for all $u \in \mathcal{U}^c$, $\varphi_i \circ \varphi(u_i, z, \sigma) = \vartheta_i(u_{i1}) + \eta_i(z, \sigma)$.

The following result provides sufficient conditions for some of our axioms to be satisfied.

Proposition 2. *Assume R, R^p and R^a are defined as in Proposition 1. Then they satisfy Axioms 1 (Ordering), 2 (Dominance), 3 (Independence), 5 (State Neutrality), 6 (State Equivalence), 7 (Ex Post Pareto), 10 (Independence of the Utility of the Sure), and the first part of Axiom 8 (Monotonicity).*

Note that we do not provide necessary and sufficient conditions for all the axioms to be satisfied. This is because Axiom 4 (Ex Post Richness) and the second part of Axiom 8 (Monotonicity) impose conditions on the functions introduced in the above proposition which are rather heavy to write down. However, we can provide the following example where all the axioms are satisfied:

Example 1. R, R^p and R^a are defined as follows:

1. There exist $\alpha \neq 0, \lambda \in (0, 1)$, an increasing function f and a function π such that for all $u, u' \in \mathcal{U}, z, z' \in \mathcal{Z}, \sigma, \sigma' \in \mathcal{S}$,
$$\begin{aligned}(u_i, z, \sigma) R \left(u'_i, z', \sigma'\right) &\Leftrightarrow \frac{1}{\alpha} \ln(\pi(z, \sigma)) + f\left(\lambda u_{i\sigma} + (1-\lambda) \sum_{s \in \mathcal{S}} \pi(z, s) u_{is}\right) \\ &\geq \frac{1}{\alpha} \ln(\pi(z', \sigma')) + f\left(\lambda u'_{i\sigma'} + (1-\lambda) \sum_{s \in \mathcal{S}} \pi(z', s) u'_{is}\right)\end{aligned}$$
where $\sum_{s \in \mathcal{S}} \pi(z, s) = 1$ for all $z \in \mathcal{Z}$ and $\pi(z^*, s) = 1/s$ for all $s \in \mathcal{S}$;
2. For all $u, u' \in \mathcal{U}, z, z' \in \mathcal{Z}, \sigma, \sigma' \in \mathcal{S}$,
$$\begin{aligned}(u, z, \sigma) R^p (u', z', \sigma') &\Leftrightarrow \frac{1}{\alpha} \ln(\pi(z, \sigma)) + \frac{1}{n} \sum_{i \in \mathcal{N}} f\left(\lambda u_{i\sigma} + (1-\lambda) \sum_{s \in \mathcal{S}} \pi(z, s) u_{is}\right) \\ &\geq \frac{1}{\alpha} \ln(\pi(z', \sigma')) + \frac{1}{n} \sum_{i \in \mathcal{N}} f\left(\lambda u'_{i\sigma'} + (1-\lambda) \sum_{s \in \mathcal{S}} \pi(z', s) u'_{is}\right).\end{aligned}$$
3. For all $u, u' \in \mathcal{U}$, $z, z' \in \mathcal{Z}$,
$$\begin{aligned}(u, z) R^a (u', z') &\Leftrightarrow \sum_{\sigma \in \mathcal{S}} \alpha \pi(z, \sigma) \exp\left(\frac{\alpha}{n} \sum_{i \in \mathcal{N}} f\left(\lambda u_{i\sigma} + (1-\lambda) \sum_{s \in \mathcal{S}} \pi(z, s) u_{is}\right)\right) \\ &\geq \sum_{\sigma \in \mathcal{S}} \alpha \pi(z', \sigma) \exp\left(\frac{\alpha}{n} \sum_{i \in \mathcal{N}} f\left(\lambda u'_{i\sigma} + (1-\lambda) \sum_{s \in \mathcal{S}} \pi(z', s) u'_{is}\right)\right).\end{aligned}$$

While the additive family (a) in Proposition 1 is reminiscent of Harsanyi's result, the exponential one (b) looks more singular. However, when Axioms 9 (Anonymity) and 11 (Restricted Independence of the Sure) are added, only the additive family remains.

Proposition 3. *If Axioms 1 (Ordering), 2 (Dominance), 3 (Independence), 4 (Ex Post Richness), 5 (State Neutrality), 6 (State Equivalence), 7 (Ex Post Pareto), 8 (Monotonicity), 9 (Anonymity), 10 (Independence of the Utility of the Sure) and 11 (Restricted Independence of the Sure) are satisfied then there exists a continuous function $\varphi : \mathbb{R}^s \times \mathcal{Z} \times \mathcal{S} \to \mathbb{R}$ that is increasing in its first s arguments and satisfies $\varphi(u_i, z^*, \sigma) = \varphi(u_i, z^*, \sigma')$ for all $\sigma, \sigma' \in \mathcal{S}$ and $u \in \mathcal{U}^c$,* *and* $\sum_{\sigma\in\mathcal{S}}\varphi(u_i,z,\sigma)=\psi(u_i^1)+\xi(z)$ *for all* $u\in\mathcal{U}^c$*, where* ψ *is continuous and increasing and* ξ *is continuous, and such that:*

1. *For all* $u_i, u_i'\in\mathbb{R}^s,\ z,z'\in\mathcal{Z},\ \sigma,\sigma'\in\mathcal{S}$*,*
$$(u_i,z,\sigma)\,R\left(u_i',z',\sigma'\right)\Leftrightarrow\varphi\left(u_i,z,\sigma\right)\geq\varphi\left(u_i',z',\sigma'\right).$$
2. *For all* $u,u'\in\mathcal{U},\ z,z'\in\mathcal{Z},\ \sigma,\sigma'\in\mathcal{S}$*,*
$$(u,z,\sigma)\,R^p\left(u',z',\sigma'\right)\Leftrightarrow\sum_{i\in\mathcal{N}}\varphi\left(u_i,z,\sigma\right)\geq\sum_{i\in\mathcal{N}}\varphi\left(u_i',z',\sigma'\right).$$
3. *For all* $u,u'\in\mathcal{U},\ z,z'\in\mathcal{Z}$*,*
$$(u,z)R^a(u',z')\Leftrightarrow\sum_{i\in\mathcal{N}}\sum_{\sigma\in\mathcal{S}}\varphi\left(u_i,z,\sigma\right)\geq\sum_{i\in\mathcal{N}}\sum_{\sigma\in\mathcal{S}}\varphi\left(u_i',z',\sigma\right).$$

5. Discussion

Let us review the various possible social criteria that are singled out or ruled out by our results. In order to do so, it is useful to specify a little more the notion of individual ex ante utility. We will assume that it can be denoted E_zu_i, without implying that this must be an *expected* utility.

In comparison with the more restrictive analysis in Fleurbaey (2010), allowing for a richer evaluation of final consequences that takes account of counterfactual states makes it possible here to combine ex ante inequality aversion with ex post rationality, separability, and some respect of ex ante utility. The introduction of ex post inequality aversion remains however problematic.

Consider the criterion based on $\sum_{i\in\mathcal{N}}\sum_{\sigma\in\mathcal{S}}\varphi\left(u_i,z,\sigma\right)$. If $u\in\mathcal{U}^e$, i.e., if u is perfectly egalitarian in all states of nature, the criterion boils down to maximizing $\sum_{\sigma\in\mathcal{S}}\varphi\left(u_i,z,\sigma\right)$. If this formula is congruent with individual ex ante utility (which may be considered appealing when there is no inequality), there is an increasing function G such that $\sum_{\sigma\in\mathcal{S}}\varphi\left(u_i,z,\sigma\right)\equiv G\left(E_zu_i\right)$. This implies that for all $u\in\mathcal{U}$, not just egalitarian prospects, the social criterion is based on $\sum_{i\in\mathcal{N}}G\left(E_zu_i\right)$. It is clear that this criterion takes no account whatsoever of ex post inequalities and focuses at most on ex ante inequalities. Therefore Proposition 3 implies that there is a clear dilemma between respecting ex ante individual utility in absence of inequalities and giving priority to the worst-off in every state of nature.

Let us now move backward and examine another possibility left open in Proposition 1 and excluded in Proposition 3, namely, the case in which R^a is represented by

$$\alpha\sum_{\sigma\in\mathcal{S}}\exp\left(\alpha\sum_{i\in\mathcal{N}}\varphi\left(u_i,z,\sigma\right)\right).$$

To ensure Independence of the Utility of the Sure (see Proposition 1), we can use the model proposed in Example 1, namely

$$\varphi\left(u_i,z,\sigma\right)=\frac{1}{n}\Big(\frac{1}{\alpha}\ln(\pi(z,\sigma))+f(\lambda u_{i\sigma}+(1-\lambda)E_zu_i)\Big),$$

for $E_zu_i=\sum_{s\in\mathcal{S}}\pi(z,s)u_{is}$, so that the ex ante criterion is

$$\alpha\sum_{\sigma\in\mathcal{S}}\pi(z,\sigma)\exp\left(\alpha\sum_{i\in\mathcal{N}}\frac{1}{n}f\Big(\lambda u_{i\sigma}+(1-\lambda)E_zu_i\Big)\right).$$

This criterion embodies a concern for ex ante and ex post inequality as well as for ex ante fairness. The cost is of course to lose Restricted Independence of the Sure. For specific functions f, the criterion can also respect ex ante individual judgments in the absence of inequality. When $\alpha>0$, the expression

$$\alpha\sum_{\sigma\in\mathcal{S}}\pi(z,\sigma)\exp\Big(\alpha f\Big(\lambda u_{i\sigma}+(1-\lambda)E_zu_i\Big)\Big)$$

is indeed a function of E_zu_i provided $f(y)=\frac{\ln(y+\gamma)}{\alpha}$. This criterion is however well defined only when $\lambda u_{i\sigma}+(1-\lambda)E_zu_i>-\gamma$. Hence the dilemmas exposed above cannot be solved satisfactorily on the whole space $\mathcal{U}$ even when relaxing Restricted Independence of the Sure.

6. Conclusion

The general outlook of our results is negative. In a framework that allows for an explicit incorporation of ex ante and ex post inequality aversion and ex ante fairness in the social evaluation of risky prospects, a list of standard or seemingly mild axioms that capture basic notions of social rationality, individualism and separability impose such constraints on the social ordering that no fully satisfactory candidate is singled out. Ex post inequality aversion enters in conflict with a minimal respect of individual ex ante utility. The conflict is alleviated only if separability is relaxed.

A more positive conclusion is that we have found general conditions that single out an additively separable criterion of the form $\sum_{i\in\mathcal{N}}\sum_{\sigma\in\mathcal{S}}\varphi\left(u_i,z,\sigma\right)$ without assuming a strict form of consequentialism. Although we have directly assumed separability across states of nature via the Independence axiom, separability across subpopulations has been introduced only through weak separability axioms involving the subpopulations who take no risk. Another positive result is the exponential criterion which, even though it fails some attractive axioms, illustrates how a specific degree of inequality aversion may be imposed by the analysis.

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Appendix. Proofs

A.1. Proof of Lemma 1

Proof. Let $u,v,u',v'\in\mathcal{U},z,z'\in\mathcal{Z},\sigma,\sigma'\in\mathcal{S}$, and $M\subset\mathcal{N}$ be such that

$$\forall i\in M,\quad u_i=v_i$$
$$\forall i\in M,\quad u_i'=v_i'$$
$$\forall i\in\mathcal{N}\setminus M,\quad (u_i,z,\sigma)I(u_i',z',\sigma')$$
$$\forall i\in\mathcal{N}\setminus M,\quad (v_i,z,\sigma)I(v_i',z',\sigma').$$

By Axioms 1 (Ordering) and 8 (Monotonicity), for all $i\in\mathcal{N}$ there is a sure $\bar{u}_i$ such that $(u_i,z,\sigma)\,I\,(\bar{u}_i,z^*,\sigma)$ and a sure $\bar{u}_i'$ such that $\left(u_i',z',\sigma'\right)I\left(\bar{u}_i',z^*,\sigma'\right)$. Similarly there is a sure $\bar{v}_i$ such that $(v_i,z,\sigma)\,I\,(\bar{v}_i,z^*,\sigma)$ and a sure $\bar{v}_i'$ such that $\left(v_i',z',\sigma'\right)I\left(\bar{v}_i',z^*,\sigma'\right)$. For all $i\in M$, $\bar{u}_i=\bar{v}_i$ and $\bar{u}_i'=\bar{v}_i'$. For all $i\in\mathcal{N}\setminus M$, $(\bar{u}_i,z^*,\sigma)\,I\left(\bar{u}_i',z^*,\sigma'\right)$ and $(\bar{v}_i,z^*,\sigma)\,I\left(\bar{v}_i',z^*,\sigma'\right)$, which, by Axiom 6 (State Equivalence), implies $\bar{u}_i=\bar{u}_i'$ and $\bar{v}_i=\bar{v}_i'$.

By Axiom 7 (Ex Post Pareto),

$$(u,z,\sigma)\,I^p\left(\bar{u},z^*,\sigma\right),$$
$$\left(u',z',\sigma'\right)I^p\left(\bar{u}',z^*,\sigma'\right),$$
$$(v,z,\sigma)\,I^p\left(\bar{v},z^*,\sigma\right),$$
$$\left(v',z,\sigma\right)I^p\left(\bar{v}',z^*,\sigma\right).$$

Suppose that $(u, z, \sigma)R^p\left(u', z', \sigma'\right)$, which, by transitivity, is equivalent to $(\bar{u}, z^*, \sigma)\, R^p \left(\bar{u}', z^*, \sigma'\right)$. By Axiom 6 (State Equivalence), $(\bar{u}, z^*, \tau)\, R^p \left(\bar{u}', z^*, \tau\right)$ for all $\tau \in \mathcal{S}$. By Axiom 2 (Dominance), $(\bar{u}, z^*)\, R^a \left(\bar{u}', z^*\right)$.

By Axiom 10 (Independence of Utility of the Sure),

$$\left(\left(\bar{u}_M, \bar{u}_{\mathcal{N}\setminus M}\right), z^*\right) R^a \left(\left(\bar{u}'_M, \bar{u}_{\mathcal{N}\setminus M}\right), z^*\right) \Leftrightarrow$$
$$\left(\left(\bar{u}_M, \bar{u}'_{\mathcal{N}\setminus M}\right), z^*\right) R^a \left(\left(\bar{u}'_M, \bar{u}'_{\mathcal{N}\setminus M}\right), z^*\right).$$

This also reads

$$\left(\bar{u}, z^*\right) R^a \left(\bar{u}', z^*\right) \Leftrightarrow \left(\bar{v}, z^*\right) R^a \left(\bar{v}', z^*\right).$$

Suppose one had $\left(\bar{v}', z^*, \sigma'\right) P^p \left(\bar{v}, z^*, \sigma\right)$. Then, by Axioms 6 (State Equivalence) and 2 (Dominance), one would have $\left(\bar{v}', z^*\right) P^a \left(\bar{v}, z^*\right)$, a contradiction. Therefore, $(\bar{v}, z^*, \sigma)\, R^p \left(\bar{v}', z^*, \sigma'\right)$. By transitivity, $(v, z, \sigma)\, R^p(v', z', \sigma')$.

We have proved that

$$(u, z, \sigma)R^p\left(u', z', \sigma'\right) \Rightarrow (v, z, \sigma)\, R^p(v', z', \sigma').$$

By symmetry, the converse holds. □

A.2. Proof of Proposition 1

Claim 1. *There exists a continuous function $\varphi : \mathbb{R}^s \times \mathcal{Z} \times \mathcal{S} \to \mathbb{R}$, increasing in its first s arguments and satisfying $\varphi(u_i, z^*, \sigma) = \varphi(u_i, z^*, \sigma')$ for all $\sigma, \sigma' \in \mathcal{S}$ and $u \in \mathcal{U}^c$, such that for all $u_i, u'_i \in \mathbb{R}^s$, $z, z' \in \mathcal{Z}, \sigma, \sigma' \in \mathcal{S}$,*

$$(u_i, z, \sigma)\, R\left(u'_i, z', \sigma'\right) \Leftrightarrow \varphi\left(u_i, z, \sigma\right) \geq \varphi\left(u'_i, z', \sigma'\right).$$

Proof. By Axiom 1 (Ordering), there is a real-valued continuous function φ that represents R. By Axiom 8 (Monotonicity), it is increasing in each component of u_i. By Axiom 6 (State Equivalence), it must be the case that $\varphi(u_i, z^*, \sigma) = \varphi(u_i, z^*, \sigma')$ for all $\sigma, \sigma' \in \mathcal{S}$ and $u \in \mathcal{U}^c$. □

Claim 2. *There exist n continuous increasing functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ such that for all $u, u' \in \mathcal{U}, z, z' \in \mathcal{Z}, \sigma, \sigma' \in \mathcal{S}$,*

$$(u, z, \sigma)R^p(u', z', \sigma') \Leftrightarrow \sum_{i\in\mathcal{N}} \varphi_i \circ \varphi(u_i, z, \sigma) \geq \sum_{i\in\mathcal{N}} \varphi_i \circ \varphi(u'_i, z', \sigma').$$

Proof. By Axioms 1 (Ordering) and 7 (Ex Post Pareto), there is a continuous and increasing function[12] $\Gamma : (\text{rge}\,\varphi)^n \to \mathbb{R}$ such that for all $u, u' \in \mathcal{U},\ z, z' \in \mathcal{Z}, \sigma, \sigma' \in \mathcal{S}$,

$$(u, z, \sigma)R^p(u', z', \sigma') \Leftrightarrow \Gamma((\varphi(u_i, z, \sigma))_{i\in\mathcal{N}}) \geq \Gamma((\varphi(u'_i, z', \sigma'))_{i\in\mathcal{N}}).$$

By Lemma 1, Axiom 12 (Ex Post Separability) holds, so that the ordering over $(\text{rge}\,\varphi)^n$ represented by Γ is separable. Therefore, there exist n continuous functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ such that for all $u, u' \in \mathcal{U}, z, z' \in \mathcal{Z},\ \sigma, \sigma' \in \mathcal{S}$,

$$(u, z, \sigma)R^p(u', z', \sigma') \Leftrightarrow \sum_{i\in\mathcal{N}} \varphi_i \circ \varphi(u_i, z, \sigma) \geq \sum_{i\in\mathcal{N}} \varphi_i \circ \varphi(u'_i, z', \sigma').$$

By Axiom 7 (Ex Post Pareto), each φ_i is increasing. □

Claim 3. *There exists a continuous and increasing function $\Psi : \mathbb{R} \to \mathbb{R}$ such that for all $u, u' \in \mathcal{U},\ z, z' \in \mathcal{Z}$,*

$$(u, z)R^a(u', z') \Leftrightarrow \sum_{\sigma\in\mathcal{S}} \Psi\left(\sum_{i\in\mathcal{N}} \varphi_i \circ \varphi(u_i, z, \sigma)\right) \geq \sum_{\sigma\in\mathcal{S}} \Psi\left(\sum_{i\in\mathcal{N}} \varphi_i \circ \varphi(u'_i, z', \sigma')\right).$$

Proof. By Axiom 4 (Ex Post Richness), for any $z \in \mathcal{Z}$,

$$\Gamma_z = \{(\varphi(u, z, \sigma))_{\sigma\in\mathcal{S}} | u \in \mathcal{U}\} = \prod_{\sigma\in\mathcal{S}} \text{rge}\,\varphi(\cdot, z, \sigma).$$

Define $\Gamma = \{(\varphi(u, z, \sigma))_{\sigma\in\mathcal{S}} | (u, z) \in \mathcal{U} \times \mathcal{Z}\}$, so that $\Gamma = \cup_{z\in\mathcal{Z}} \Gamma_z$. Also define $\Upsilon_\sigma = \text{rge}\,\varphi(\cdot, z^*, \sigma)$. By part 2 of Axiom 8 (Monotonicity), and the product structure of Γ_{z^*}, $\Gamma_z \subset \Gamma_{z^*}$ so that $\Gamma = \Gamma_{z^*} = \prod_{\sigma\in\mathcal{S}} \Upsilon_\sigma$.

We let $\psi(u, z, \sigma) = \sum_{i\in\mathcal{N}} \varphi_i \circ \varphi(u_i, z, \sigma)$ and define

$$\mathcal{D} = \{(\psi(u, z, \sigma))_{\sigma\in\mathcal{S}} | (u, z) \in \mathcal{U} \times \mathcal{Z}\}.$$

The structure of Γ implies that $\mathcal{D} = \prod_{\sigma\in\mathcal{S}} D_\sigma$, where

$$D_\sigma = \left\{d_\sigma \in \mathbb{R} | \exists (d_i)_{i\in\mathcal{N}} \in \Upsilon_\sigma^n, d_\sigma = \sum_{i\in\mathcal{N}} \varphi_i(d_i)\right\}.$$

Define $\succeq$ on $\mathcal{D}$ as follows: $a \succeq b$ iff there exist $u, u' \in \mathcal{U}, z, z' \in \mathcal{Z}$,

$$\left.\begin{array}{ll} \forall\sigma \in \mathcal{S}, & \psi(u, z, \sigma) = a_\sigma \\ \forall\sigma \in \mathcal{S}, & \psi(u', z', \sigma) = b_\sigma \end{array}\right\} \quad \text{and} \quad (u, z)R^a(u', z').$$

By Axioms 1 (Ordering), 2 (Dominance) and 4 (Ex Post Richness), $\succeq$ is a well-defined, complete and continuous ordering, as we now show.

First, observe that $\succeq$ is well-defined. Indeed, let $u, v, u', v' \in \mathcal{U}$ and $y, z, y', z' \in \mathcal{Z}$ be such that for all $\sigma \in \mathcal{S}$, $\psi(u, z, \sigma) = \psi(v, y, \sigma) = a_\sigma$ and $\psi(u', z', \sigma) = \psi(v', y', \sigma) = b_\sigma$. This implies that for all $\sigma \in \mathcal{S}$, $(u, z, \sigma)I^p(v, y, \sigma)$ and $(u', z', \sigma)I^p(v', y', \sigma)$. By Axiom 2 (Dominance), $(u, z)I^a(v, y)$ and $(u', z')I^a(v', y')$. Therefore,

$$(u, z)R^a(u', z') \Leftrightarrow (v, y)R^a(v', y').$$

Since $\mathcal{D} = \{(\psi(u, z, \sigma))_{\sigma\in\mathcal{S}} | (u, z) \in \mathcal{U} \times \mathcal{Z}\}$ and R^a is complete, $\succeq$ is also complete.

Finally, we show that $\succeq$ is continuous. The function $\bar{\psi} : \mathcal{U} \times \mathcal{Z} \to \mathcal{D}$ defined by $\bar{\psi}(u, z) = (\psi(u, z, \sigma))_{\sigma\in\mathcal{S}}$ is continuous. Consider any $b \in \mathcal{D}$ and the set $\mathcal{A} = \{a \in \mathcal{D} | a \succeq b\}$. Let v, y be such that $\bar{\psi}(v, y) = b$. The set $\mathcal{A}$ is the image by $\bar{\psi}$ of the set

$$\left\{(u, z) \in \mathcal{U} \times \mathcal{Z} | (u, z)R^a(v, y)\right\}.$$

As R^a is continuous, this set is closed. Since $\bar{\psi}$ is continuous, $\mathcal{A}$ is also closed. A similar argument shows that the set $\{a \in \mathcal{D} | b \succeq a\}$ is closed as well, and therefore $\succeq$ is continuous.

By Axiom 2 (Dominance), $\succeq$ is strictly monotonic in each component. By Axiom 3 (Independence), it is separable. By Axiom 5 (State Neutrality), it is symmetric, i.e., indifferent to permutations of components.

Therefore there exists a continuous and increasing function $\Psi : \mathbb{R} \to \mathbb{R}$ such that for all $a, b \in \mathcal{D}$,

$$a \succeq b \Leftrightarrow \sum_{\sigma\in\mathcal{S}} \Psi(a_\sigma) \geq \sum_{\sigma\in\mathcal{S}} \Psi(b_\sigma). \quad \square$$

Claim 4. *One can restrict attention either to $\Psi(x) = x$ or to $\Psi(x) = \alpha e^{\alpha x}$ for some $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof. Let $\{C, R\}$ be a partition of $\mathcal{N}$, with $|C| \geq 2$. Let $\mathcal{U}_C^c \subset U$ be the subset of matrices u such that u_C is risk-free. Finally, let $r = |R|$.

[12] For any function f, rge f denotes the range of f.

For $i \in \mathcal{N}$ and $x \in \mathbb{R}$, let $\phi_i(x) = \varphi_i \circ \varphi((x, \dots, x), z^*, \sigma)$. Note that by Axiom 6 (State Equivalence) this value does not depend on σ. Each function ϕ_i is continuous and increasing. Without loss of generality, we can impose $\phi_i(0) = 0$.

By Axiom 10 (Independence of the Utility of the Sure), for R^a the subset $R \cup A$ is separable for all $A \subsetneq C$ (including $A = \emptyset$). Therefore, by Theorem 1 in Gorman (1968), every subset of C, including C itself, is also separable. By corollary of Theorem 1 in Gorman (1968), there exist continuous functions $h : \mathbb{R}^{rs} \to \mathbb{R}$ and $\hat{\phi}_i : \mathbb{R} \to \mathbb{R}$, $i \in C$, such that for all $u, v \in \mathcal{U}^c_C$,

$$(u, z^*)R^a(v, z^*) \Leftrightarrow h(u_R) + \sum_{i\in C} \hat{\phi}_i(u_{i1}) \geq h(v_R) + \sum_{i\in C} \hat{\phi}_i(v_{i1}).$$

Therefore, there exists an increasing function $\bar{f}$ such that for all $u \in \mathcal{U}^c_C$,

$$\sum_{\sigma\in\mathcal{S}} \Psi\left(\sum_{i\in R} \varphi_i \circ \varphi(u_i, z^*, \sigma) + \sum_{i\in C} \phi_i(u_{i1})\right) = \bar{f}\left(h(u_R) + \sum_{i\in C} \hat{\phi}_i(u_{i1})\right). \quad (1)$$

Fixing u_R, one sees that this implies that there is an increasing function $\bar{g}$ such that for all u_C such that $(u_R, u_C) \in \mathcal{U}^c_C$,

$$\sum_{i\in C} \hat{\phi}_i(u_{i1}) = \bar{g}\left(\sum_{i\in C} \phi_i(u_{i1})\right).$$

Moreover, letting $\phi^*_i = \hat{\phi}_i \circ \phi_i^{-1}$, this reads

$$\sum_{i\in C} \phi^*_i(\phi_i(u_{i1})) = \bar{g}\left(\sum_{i\in C} \phi_i(u_{i1})\right).$$

Using the independent variables $w_i = \phi(u_{i1})$, we obtain a variant of a Pexider equation on the non-empty rectangular set[13] $W = \left\{(w_i)_{i\in C} : \forall i\ \exists u_i \in \mathbb{R} \text{ s.t. } x_i = \phi_i(u_i)\right\}$. This implies that $\bar{g}$ and ϕ^*_i must be affine (Radó and Baker, 1987), so that there exist γ, δ such that $\phi_i(u_{i1}) = \gamma\hat{\phi}_i(u_{i1}) + \delta_i$. Remark that the functions ϕ and $\hat{\phi}$ are defined before u_R is fixed so that the scalars γ and δ_i are independent of u_R.

As a result, one can simplify Eq. (1) and write that there exists an increasing function f such that for all $u \in \mathcal{U}^c_C$,

$$\sum_{\sigma\in\mathcal{S}} \Psi\left(\sum_{i\in R} \varphi_i \circ \varphi(u_i, z^*, \sigma) + \sum_{i\in C} \phi_i(u_{i1})\right) = f\left(h(u_R) + \sum_{i\in C} \phi_i(u_{i1})\right). \quad (2)$$

For $u_C = \mathbf{0_C}$, this implies

$$h(u_R) = f^{-1}\left(\sum_{\sigma\in\mathcal{S}} \Psi\left(\sum_{i\in R} \varphi_i \circ \varphi(u_i, z^*, \sigma)\right)\right).$$

Substituting in Eq. (2), we obtain

$$\sum_{\sigma\in\mathcal{S}} \Psi\left(\sum_{i\in R} \varphi_i \circ \varphi(u_i, z^*, \sigma) + \sum_{i\in C} \phi_i(u_{i1})\right) = f\left(f^{-1}\left(\sum_{\sigma\in\mathcal{S}} \Psi\left(\sum_{i\in R} \varphi_i \circ \varphi(u_i, z^*, \sigma)\right)\right) + \sum_{i\in C} \phi_i(u_{i1})\right). \quad (3)$$

Fix u_c and define $t = \sum_{i\in C} \phi_i(u_{i1})$ and $x_\sigma = \Psi\left(\sum_{i\in R} \varphi_i \circ \varphi(u_i, z^*, \sigma)\right)$. Let $X = \left\{(x_\sigma)_{\sigma\in\mathcal{S}} : \forall\sigma \in \mathcal{S}\ \exists u \in \mathcal{U} \text{ s.t. } x_\sigma = \Psi\left(\sum_{i\in R} \varphi_i \circ \varphi(u_i, z^*, \sigma)\right)\right\}$. We know from the proof of Claim 3 that

$$\Gamma_{z^*} = \left\{(\varphi(u, z^*, \sigma))_{\sigma\in\mathcal{S}} | u \in \mathcal{U}\right\} = \prod_{\sigma\in\mathcal{S}} \Upsilon_\sigma.$$

Defining

$$X_\sigma = \left\{d_\sigma \in \mathbb{R} | \exists (d_i)_{i\in\mathcal{N}} \in \Upsilon^n_\sigma,\ d_\sigma = \Psi\left(\sum_{i\in\mathcal{N}} \varphi_i(d_i)\right)\right\}$$

one obtains $X = \prod_{\sigma\in\mathcal{S}} X_\sigma$.

Equality (3) reads:

$$\sum_{\sigma\in\mathcal{S}} \Psi\left(\Psi^{-1}(x_\sigma) + t\right) = f\left(f^{-1}\left(\sum_{\sigma\in\mathcal{S}} x_\sigma\right) + t\right).$$

This is a Pexider equation defined on X. The set X has a connected non-empty interior, so that the Corollary 3 in Radó and Baker (1987) applies: there exist $\gamma(t) > 0$, $\delta(t)$ such that

$$\Psi\left(\Psi^{-1}(x_\sigma) + t\right) = \gamma(t)x_\sigma + \delta(t).$$

Equivalently, letting $y_\sigma = \Psi^{-1}(x_\sigma) = \sum_{i\in R} \varphi_i \circ \varphi(u_i, z^*, \sigma)$,

$$\Psi(y_\sigma + t) = \gamma(t)\Psi(y_\sigma) + \delta(t).$$

By Corollary 1 (pp. 150–151) in Aczél (1966) this equation implies that $\Psi(x)$ is affine in x or affine in $\alpha e^{\alpha x}$ for some $\alpha \neq 0$. □

Claim 5. *For all $u \in \mathcal{U}^c$ and $i \in \mathcal{N}$ it must be the case that*

1. *when $\Psi(x) = x$, $\sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(u, z, \sigma) = \psi_i(u_{i1}) + \xi_i(z)$ where ψ_i is continuous and increasing and ξ_i is continuous.*
2. *when $\Psi(x) = \alpha e^{\alpha x}$, $\varphi_i \circ \varphi(u, z, \sigma) = \vartheta_i(u_{i1}) + \eta_i(z, \sigma)$ where ϑ_i is continuous and increasing and η_i is continuous.*

Proof. *Case* 1: $\Psi(x) = x$. Let $u, u' \in \mathcal{U}$, $v, v' \in \mathcal{U}^c$, $z \in \mathcal{Z}$ and $z_0 \in \mathcal{Z}$ a reference informational content (for instance, equiprobable states of the world). By Axiom 10 (Independence of Utility of the Sure), it must be the case that:

$$((u_{-i}, v_i), z)R^a((u'_{-i}, v_i), z_0) \Leftrightarrow ((u_{-i}, v'_i), z)R^a((u'_{-i}, v'_i), z_0).$$

Using the representation in Case 1, this means that:

$$\begin{aligned}&\sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v_i, z, \sigma) + \sum_{j\neq i}\sum_{\sigma\in\mathcal{S}} \varphi_j \circ \varphi(u_j, z, \sigma)\\ &\geq \sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v_i, z_0, \sigma) + \sum_{j\neq i}\sum_{\sigma\in\mathcal{S}} \varphi_j \circ \varphi(u'_j, z_0, \sigma)\\ &\Leftrightarrow \sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v'_i, z, \sigma) + \sum_{j\neq i}\sum_{\sigma\in\mathcal{S}} \varphi_j \circ \varphi(u_j, z, \sigma)\\ &\geq \sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v'_i, z_0, \sigma) + \sum_{j\neq i}\sum_{\sigma\in\mathcal{S}} \varphi_j \circ \varphi(u'_j, z_0, \sigma).\end{aligned}$$

Hence the difference $\sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v_i, z, \sigma) - \sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v_i, z_0, \sigma)$ is independent of v_i: there exists a function ξ_i such that $\sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v_i, z, \sigma) - \sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v_i, z_0, \sigma) = \xi_i(z)$. Denoting $\psi_i(v_{i1}) = \sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v_i, z_0, \sigma)$ yields the result. Axioms 1 (Ordering) and 8 (Monotonicity) imply the properties of ψ_i and ξ_i.

Case 2: $\Psi(x) = \alpha e^{\alpha x}$. Let $u, u' \in \mathcal{U}$, $v \in \mathcal{U}^c$, $z \in \mathcal{Z}$. Let $\mathbf{0}$ denotes the sure prospect in $\mathbb{R}^s$ with all its components equal to 0. By Axiom 10 (Independence of Utility of the Sure), it must be the case that:

$$((u_{-i}, v_i), z)R^a((u'_{-i}, v_i), z) \Leftrightarrow ((u_{-i}, \mathbf{0}), z)R^a((u'_{-i}, \mathbf{0}), z).$$

[13] This is so because the functions ϕ_i are continuous and increasing.

Using the representation in Case 2, and assuming without loss of generality that $\alpha > 0$. This means that:

$$\sum_{\sigma\in\mathcal{S}} \exp\Big(\alpha\varphi_i \circ \varphi(v_i, z, \sigma)\Big) \exp\Big(\alpha \sum_{j\neq i} \varphi_j \circ \varphi(u_j, z, \sigma)\Big)$$
$$\geq \sum_{\sigma\in\mathcal{S}} \exp\Big(\alpha\varphi_i \circ \varphi(v_i, z, \sigma)\Big) \exp\Big(\alpha \sum_{j\neq i} \varphi_j \circ \varphi(u'_j, z, \sigma)\Big)$$
$$\Leftrightarrow \sum_{\sigma\in\mathcal{S}} \exp\Big(\alpha\varphi_i \circ \varphi(\mathbf{0}, z, \sigma)\Big) \exp\Big(\alpha \sum_{j\neq i} \varphi_j \circ \varphi(u_j, z, \sigma)\Big)$$
$$\geq \sum_{\sigma\in\mathcal{S}} \exp\Big(\alpha\varphi_i \circ \varphi(\mathbf{0}, z, s)\Big) \exp\Big(\alpha \sum_{j\neq i} \varphi_j \circ \varphi(u'_j, z, \sigma)\Big).$$

Let $D_{iz} = \{(d_1, \ldots, d_s) \in \mathbb{R}^s : \exists u \in \mathcal{U}, \forall\sigma \in \mathcal{S}, d_\sigma = \sum_{j\neq i} \varphi_j \circ \varphi(u_j, z, \sigma)\}$. D_{iz} is connected and has a non empty interior. Let $f_\sigma(d_\sigma) = \exp\Big(\alpha\varphi_i \circ \varphi(v_i, z, \sigma)\Big)e^{\alpha d_\sigma}$ and $g_\sigma(d_\sigma) = \exp\Big(\alpha\varphi_i \circ \varphi(\mathbf{0}, z, \sigma)\Big)e^{\alpha d_\sigma}$. By the above result, the two representations $\sum_{\sigma\in\mathcal{S}} f_\sigma(d_\sigma)$ and $\sum_{\sigma\in\mathcal{S}} g_\sigma(d_\sigma)$ are ordinally equivalent, so that there exists a continuous and increasing function H such that $\sum_{\sigma\in\mathcal{S}} f_\sigma(d_\sigma) = H\left(\sum_{\sigma\in\mathcal{S}} g_\sigma(d_\sigma)\right)$. We obtain a Pexider equation over the range of functions g_σ: Radó and Baker (1987) results apply once again since the range is connected and non empty (functions g_σ being continuous and increasing).

There must exist $a > 0$ and scalars b_σ such that $f_\sigma = ag_\sigma + b_\sigma$. In view of the forms of functions f_σ and g_σ, we need $b_\sigma = 0$ for all $\sigma \in \mathcal{S}$.

To sum up, for every sure v_i and all $\sigma \in \mathcal{S}$, there exists $a(v_i)$ such that $\exp(\alpha\varphi_i \circ \varphi(v_i, z, \sigma)) = a(v_i) \exp(\alpha\varphi_i \circ \varphi(\mathbf{0}, z, \sigma))$. Denoting $\vartheta_i(v_{i1}) = \frac{\ln \circ a(v_i)}{\alpha}$ and $\eta_i(z, \sigma) = \varphi_i \circ \varphi(\mathbf{0}, z, \sigma)$, we obtain that $\varphi_i \circ \varphi(u, z, \sigma) = \vartheta_i(u_{i1}) + \eta_i(z, \sigma)$. The properties of ϑ_i and η_i follow from Axioms 1 (Ordering) and 8 (Monotonicity). □

A.3. Proof of Proposition 2

Proof. It is immediate to see that the defined orderings satisfy Axioms 1 (Ordering), 2 (Dominance), 3 (Independence), 5 (State Neutrality), 7 (Ex Post Pareto), and the first part of 8 (Monotonicity).

The condition $\varphi(u_i, z^*, \sigma) = \varphi(u_i, z^*, \sigma')$ for all $u \in \mathcal{U}^c$ guarantees that they satisfy Axiom 6 (State Equivalence).

For Axiom 10 (Independence of Utility of the Sure), consider $M \subset \mathcal{N}$ and $u, u' \in \mathcal{U}$, $v \in \mathcal{U}^c$, $z, z' \in \mathcal{Z}$. In the case, $\Psi(x) = x$, we obtain

$$((u_M, v_{\mathcal{N}\setminus M}), z)R^a((u'_M, v_{\mathcal{N}\setminus M}), z')$$
$$\Leftrightarrow \sum_{i\in M}\sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(u_i, z, \sigma) + \sum_{i\in\mathcal{N}\setminus M}\sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v_i, z, \sigma)$$
$$\geq \sum_{i\in M}\sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(u'_i, z', \sigma) + \sum_{i\in\mathcal{N}\setminus M}\sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(v_i, z', \sigma)$$
$$\Leftrightarrow \sum_{i\in M}\sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(u_i, z, \sigma) + \sum_{i\in\mathcal{N}\setminus M} \xi_i(z)$$
$$\geq \sum_{i\in M}\sum_{\sigma\in\mathcal{S}} \varphi_i \circ \varphi(u'_i, z', \sigma) + \sum_{i\in\mathcal{N}\setminus M} \xi_i(z').$$

This is clearly independent of v, in accordance with Axiom 10 (Independence of Utility of the Sure).

In the case $\Psi(x) = \alpha\exp(\alpha x)$, and assuming without loss of generality that $\alpha > 0$, we obtain

$$((u_M, v_{\mathcal{N}\setminus M}), z)R^a((u'_M, v_{\mathcal{N}\setminus M}), z')$$
$$\Leftrightarrow \sum_{\sigma\in\mathcal{S}} \exp\Big(\alpha \sum_{i\in M} \varphi_i \circ \varphi(u_i, z, \sigma) + \sum_{i\in\mathcal{N}\setminus M} \varphi_i \circ \varphi(v_i, z, \sigma)\Big)$$
$$\geq \sum_{\sigma\in\mathcal{S}} \exp\Big(\alpha \sum_{i\in M} \varphi_i \circ \varphi(u'_i, z', \sigma) + \sum_{i\in\mathcal{N}\setminus M} \varphi_i \circ \varphi(v_i, z', \sigma)\Big)$$
$$\Leftrightarrow \sum_{\sigma\in\mathcal{S}} \exp\Big(\alpha \sum_{i\in M} \varphi_i \circ \varphi(u_i, z, \sigma)\Big) \exp\Big(\alpha \sum_{i\in\mathcal{N}\setminus M} \eta_i(z, \sigma)\Big)$$
$$\geq \sum_{\sigma\in\mathcal{S}} \exp\Big(\alpha \sum_{i\in M} \varphi_i \circ \varphi(u'_i, z', \sigma)\Big) \exp\Big(\alpha \sum_{i\in\mathcal{N}\setminus M} \eta_i(z', \sigma)\Big).$$

This is also independent of v. □

A.4. Proof of Proposition 3

By Proposition 1, we know that there exist functions φ, $(\varphi_i)_{i\in\mathcal{N}}$ such that for all $u, u' \in \mathcal{U}$, $z, z' \in \mathcal{Z}$, $\sigma, \sigma' \in \mathcal{S}$,

$$(u, z, \sigma)R^p(u', z', \sigma') \Leftrightarrow \sum_{i\in\mathcal{N}} \varphi_i \circ \varphi(u_i, z, \sigma) \geq \sum_{i\in\mathcal{N}} \varphi_i \circ \varphi(u'_i, z', \sigma').$$

By Axiom 9 (Anonymity), one can take the $(\varphi_i)_{i\in\mathcal{N}}$ to be identical. Letting φ denote $\varphi_i \circ \varphi$, we then obtain the first two equivalences of this proposition.

By Proposition 1, we know that either for all $u, u' \in \mathcal{U}, z, z' \in \mathcal{Z}$,

$$(u, z)R^a(u', z') \Leftrightarrow \sum_{\sigma\in\mathcal{S}}\sum_{i\in\mathcal{N}} \varphi(u_i, z, \sigma) \geq \sum_{\sigma\in\mathcal{S}}\sum_{i\in\mathcal{N}} \varphi(u'_i, z', \sigma),$$

or for some $\alpha \neq 0$, for all $u, u' \in \mathcal{U}$, $z, z' \in \mathcal{Z}$,

$$(u, z)R^a(u', z') \Leftrightarrow \alpha \sum_{\sigma\in\mathcal{S}} \exp\left(\alpha \sum_{i\in\mathcal{N}} \varphi(u_i, z, \sigma)\right) \geq \alpha \sum_{\sigma\in\mathcal{S}} \exp\left(\alpha \sum_{i\in\mathcal{N}} \varphi(u'_i, z', \sigma)\right). \quad (4)$$

Suppose the latter is true, and assume without loss of generality that $\alpha > 0$. Consider $u, u' \in \mathcal{U}^e$, $v \in \mathcal{U}^c$, $M \subseteq \mathcal{N}$ with $m = |M|$ and $z \in \mathcal{Z}$. By Axiom 11 (Restricted Independence of the Sure), for all $i \in M$,

$$((u_M, v_{\mathcal{N}\setminus M}), z)R^a((u'_M, v_{\mathcal{N}\setminus M}), z)$$
$$\Leftrightarrow \alpha \sum_{\sigma\in\mathcal{S}} \exp\left(\alpha n\varphi(u_i, z, \sigma)\right) \geq \alpha \sum_{\sigma\in\mathcal{S}} \exp\left(\alpha n\varphi(u'_i, z, \sigma)\right).$$

But the left-hand side also reads, by Eq. (4),

$$\alpha \sum_{\sigma\in\mathcal{S}} \exp\left(\alpha \sum_{i\in M} \varphi(u_i, z, \sigma) + \alpha \sum_{i\in\mathcal{N}\setminus M} \varphi(u_i, z, \sigma)\right)$$
$$\geq \alpha \sum_{\sigma\in\mathcal{S}} \exp\left(\alpha \sum_{i\in M} \varphi(u'_i, z, \sigma) + \alpha \sum_{i\in\mathcal{N}\setminus M} \varphi(u'_i, z, \sigma)\right).$$

By the condition on sure prospects for the exponential case in Proposition 1, the inequality simplifies into

$$\sum_{\sigma\in\mathcal{S}} \exp\Big(\alpha(n-m)\eta(z, s)\Big) \exp(\alpha m\varphi(u_i, z, \sigma))$$
$$\geq \sum_{\sigma\in\mathcal{S}} \Big(\alpha(n-m)\eta(z, s)\Big) \exp(\alpha m\varphi(u'_i, z, \sigma)).$$

Let $X = \{(\exp(\alpha\varphi(u_i, z, \sigma)))_{\sigma\in\mathcal{S}} | u_i \in \mathbb{R}^s\}$, and $a_\sigma = \exp(\alpha\eta(z, s))$. We have obtained that for all $x, y \in X$, all $k \in \{1, \ldots, n\}$,

$$\sum_{\sigma\in\mathcal{S}} (a_\sigma)^{n-k}(x_\sigma)^k \geq \sum_{\sigma\in\mathcal{S}} (a_\sigma)^{n-k}(y_\sigma)^k$$
$$\Leftrightarrow \sum_{\sigma\in\mathcal{S}} (x_\sigma)^n \geq \sum_{\sigma\in\mathcal{S}} (y_\sigma)^n.$$

This is possible only if there is $a \in \mathbb{R}^s_{++}$ such that for all $x \in X$, there is $\lambda \in \mathbb{R}_{++}$, $x = \lambda a$. This argument applies to all $z \in \mathcal{Z}$. This implies that for all $(u_i, z) \in \mathbb{R}^s \times \mathcal{Z}$, there is $\gamma(u_i, z) \in \mathbb{R}$, $\varphi(u_i, z, \sigma) = \gamma(u_i, z) + \frac{\ln a_\sigma}{\alpha}$. The function γ must be increasing in u_i. Let $\beta_\sigma = \frac{\ln a_\sigma}{\alpha}$.

Define $\psi(u, z, \sigma) = \sum_{i \in \mathcal{N}} \varphi(u_i, z, \sigma)$. The above reasoning implies that $\psi(u, z, \sigma) = n\beta_\sigma + \sum_{i \in \mathcal{N}} \gamma(u_i, z)$. Let $\left((u^1, z^1), \ldots, (u^s, z^s)\right) \in (\mathcal{U} \times \mathcal{Z})^s$. One has, for all $\sigma \in \mathcal{S}$, $\psi(u^\sigma, z^\sigma, \sigma) = n\beta_\sigma + \sum_{i \in \mathcal{N}} \gamma(u_i^\sigma, z^\sigma)$. Axiom 4 (Ex Post Richness) requires that there is $(u, z) \in \mathcal{U} \times \mathcal{Z}$ such that for all $\sigma \in \mathcal{S}$,

$$\psi(u, z, \sigma) = n\beta_\sigma + \sum_{i \in \mathcal{N}} \gamma(u_i, z) = n\beta_\sigma + \sum_{i \in \mathcal{N}} \gamma(u_i^\sigma, z^\sigma).$$

This is possible in general only if γ is constant. This yields a contradiction.

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