Journal of Economic Theory **106**, 190–200 (2002)
doi:10.1006/jeth.2001.2851

Measuring Inequalities without Linearity in Envy: Choquet Integrals for Symmetric Capacities

Thibault Gajdos

EUREQUA, Université Paris 1 Panthéon Sorbonne, Maison des Sciences Economiques, 106-112 boulevard de l'Hôpital, 75647 Paris Cedex 13, France
gajdos@ensae.fr

Received July 18, 2000, final version received April 11, 2001; published online January 17, 2002

The (generalized) Gini indices rely on the social welfare function of a decision maker who behaves in accordance with Yaari's model, with a function f that transforms frequencies. This SWF can also be represented as the weighted sum of the welfare of all the possible coalitions in the society, where the welfare of a coalition is defined as the income of the worst-off member of that coalition. We provide a set of axioms (Ak) and prove that the three following statements are equivalent: (i) the decision maker respects (Ak), (ii) f is a polynomial of degree k, (iii) the weight of all coalitions with more than k members is equal to zero. *Journal of Economic Literature* Classification Number: D63.
Key Words: inequality measurement; Choquet integral; symmetric capacities.

1. INTRODUCTION

During the last twenty years, a large literature has been devoted to various generalization of the Gini index (see e.g. Donaldson and Weymark [7], Weymark [14], Yaari [15], Ebert [8], Chakravarty [5], Bossert [3]). Although these works help to understand the normative basis of the Gini indices, it may be difficult to choose a specific index among the wide class of generalized Gini indices. The aim of this paper is to provide a normative benchmark for such a choice.

The Gini index and its generalizations are known to rely on non-additive expected utility functionals, and more precisely on Choquet integrals with respect to some non-additive measures (see e.g. Yaari [16], Ben-Porath and Gilboa [1], Grabisch [11]). As pointed out by Gilboa and Schmeidler

0022-0531/02 $35.00

[9], these non-additive functionals admit a "canonical representation", that has a nice interpretation in the context of social choice. Let the society be described as the set of all the possible coalitions in that society. Define the utility level of a coalition as the utility level of its worst-off member. Then, the Choquet integral may be written as a linear combination of these coalitional utility levels.

But, actually, the Gini index and its generalizations are related to a rather particular case of Choquet integrals: the non-additive measures used here are symmetric. Which means that one can write the Choquet integral in a rather simple way, using a frequency distortion function. This paper focuses on the link between the canonical representation of a Choquet integral with respect to a symmetric non-additive measure, the distortion function and the Gini index that relies on this Choquet integral.

More specifically, we prove a result suggested by Gilboa and Schmeidler [10], namely that the weight of all coalitions with more than k members is equal to zero if and only if the distortion function is a polynomial of degree (at most) k. Now, assume that a decision maker behaves in accordance with Yaari's dual model (so the inequality index that relies on this decision maker's preference is a generalized Gini index). We provide a set of axioms (Ak) that generalize an axiom proposed by Ben-Porath and Gilboa [1], such that the decision maker respects axiom (Ak) if and only if his frequency distortion function is a polynomial of degree (at most) k. We call P-Gini the generalized Gini indices with a polynomial distortion function. These indices have a very natural interpretation in terms of the weight the decision maker puts on envy in the society. These results may provide a normative benchmark for the choice of a specific generalized Gini index among the wide class of all the generalized Ginis.

2. CANONICAL REPRESENTATION, SYMMETRIC CAPACITIES AND POLYNOMIAL DISTORTION FUNCTIONS

Let $\Omega = \{1, ..., n\}$, with $n \in \mathbb{N}^*$ (so, n is not fixed), $\Sigma = 2^{\Omega}$ and $\Sigma' = \Sigma \setminus \{\varnothing\}$. We consider the space of random variables $G = \{g \mid g: \Omega \to \mathbb{R}\}$. A capacity is a function $v: \Sigma \to \mathbb{R}$, with $v(\varnothing) = 0$. We consider here capacities that are monotone (i.e., $A \subseteq B$ implies $v(A) \leqslant v(B)$ for all $A, B \in \Sigma$) and normalized (i.e., $v(\Sigma) = 1$). The Choquet integral of $g \in G$ with respect to v is defined as:

$$\int_C g\, dv = \int_0^{+\infty} v(\{w \mid g(\omega) \geqslant t\})\, dt + \int_{-\infty}^{0} [v(\{\omega \mid g(\omega) \geqslant t\}) - v(\Omega)]\, dt.$$

This integral can be written as (see e.g. Gilboa and Schmeidler [9])

$$\int_C g\,dv = \sum_{T \in \Sigma'} \alpha_T^v [\min_{\omega \in T} g(\omega)],$$

where

$$\alpha_T^v = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S).$$

If v is symmetric (i.e., $v(T) = v(T')$ whenever $|T| = |T'|$), then there exists a strictly increasing and continuous distortion function $f\colon [0, 1] \to [0, 1]$, with $f(0) = 0$ and $f(1) = 1$ such that $v(T) = f(P(T))$, where P is a probability measure over (Ω, Σ). Therefore, α_T^v only depends on $|T|$ and f. Hence, if $|T| = j$ we will note α_T^v as α_j^f. Furthermore, in that case, we have

$$\int_C g\,dv = \sum_{i=1}^{n} \left[f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right) \right] \tilde{g}(i),$$

where $\tilde{g}(i) := g(\pi(i))$, with π a permutation such that $\tilde{g}(1) \leqslant \tilde{g}(2) \leqslant \cdots \leqslant \tilde{g}(n)$. In other words, with a symmetric capacity, the Choquet integral reduces to Yaari's dual model [15].

A straightforward computation shows that

$$\alpha_j^f = \sum_{i=1}^{j} (-1)^{j-1} \binom{j}{i} f\left(\frac{i}{n}\right),$$

where

$$\binom{j}{i} = \frac{j!}{i!(j-i)!}.$$

Let $E\colon f(\frac{j}{n}) \mapsto f(\frac{j-1}{n})$ and $I\colon f(\frac{j}{n}) \mapsto f(\frac{j}{n})$; define $\Delta^k f(\frac{j}{n}) := (I-E)^k f(\frac{j}{n})$. A direct application of the binomial formula leads to:

$$\Delta^k f\left(\frac{j}{n}\right) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f\left(\frac{j-k+i}{n}\right).$$

Hence, we have

$$\alpha_j^j = \Delta^j f\left(\frac{j}{n}\right).$$

This relation between the weights α_j^f and the jth finite difference of f is the key of our results.

We can now state the following result.

THEOREM 2.1. *For a Choquet integral with respect to a symmetric capacity, the two following propositions are equivalent:*

(i) *f is a polynomial with degree (at most) equal to k.*
(ii) $\forall j > k$, $\alpha_j^f = 0$.

Proof. See the appendix.

Note that (i) $\Rightarrow$ (ii) had been proved in a much more general framework by Gilboa and Schmeidler [10], in the proof of their Theorem C (they don't require for Ω to be countable).

3. MEASURING INEQUALITIES WITHOUT LINEARITY IN ENVY

We denote by Γ the set of rank-ordered discrete and uniform income distributions, with values in $\mathbb{R}$. An income distribution X in Γ is defined by: $X = (x_1, \frac{1}{n}, \ldots, x_n, \frac{1}{n})$ where n belongs to $\mathbb{N}^*$, x_i belongs to $\mathbb{R}$ for all i, $x_i \leqslant x_{i+1}$ for all i in $\{1, \ldots, n-1\}$.

Let $\succcurlyeq$ be the decision maker's preference relation over Γ (as usual, we will denote $X \sim Y$ if $X \succcurlyeq Y$ and $Y \succcurlyeq X$). Assume, furthermore, that the decision maker evaluates the income distribution through a Choquet integral. Then it makes sense to require that the capacity with respect to which the Choquet integral is computed be symmetric. It may be interpreted as requiring for the decision maker to be impartial. (One can then think of the decision maker as anybody behind the veil of ignorance. Although not indisputable, this assumption is quite standard in the field of social choice and normative inequality measurement).

Therefore, the decision maker behaves in accordance with Yaari's Dual model [15]. There hence exists a strictly increasing continuous frequency distortion function $f: [0, 1] \to [0, 1]$ with $f(0) = 0$ and $f(1) = 1$, such that $\succcurlyeq$ is represented by:

$$V(X) = \sum_{i=1}^{n} \left[f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right) \right] x_i.$$

The generalized Gini index that relies on this welfare function can then be written as

$$IG(X) = 1 - \frac{\sum_{i=1}^{n} \left[f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right) \right] x_i}{\bar{X}},$$

where $\bar{X}$ denotes the mean of X.

Define $T_i^1(\varepsilon) := (0, \ldots, 0, -\varepsilon, 0, \ldots, 0) \in \Gamma$, where the $(-\varepsilon)$ term occurs in ith position, and $T_i^{k+1}(\varepsilon) := T_i^k(\varepsilon) - T_{i-1}^k(\varepsilon)$. The following table describes the first values of $T_i^k(\varepsilon)$.

	$i-l+1$	$\cdots$	$i-j$	$\cdots$	$i-4$	$i-3$	$i-3$	$i-1$	i
$k=1$									$-\varepsilon$
$k=2$								ε	$-\varepsilon$
$k=3$							$-\varepsilon$	2ε	$-\varepsilon$
$k=4$						ε	-3ε	3ε	$-\varepsilon$
$k=5$					$-\varepsilon$	4ε	-6ε	4ε	$-\varepsilon$
$\vdots$									
$k=r$	$(-1)^r\,\varepsilon$	$\cdots$	$(-1)^{j+1}\binom{r-1}{j}\varepsilon$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$r\varepsilon$	$-\varepsilon$

Consider the following set of axioms:

Axiom (Ak). $\forall X \in \Gamma$, $X + T_i^k(\varepsilon) \sim X + T_j^k(\varepsilon)$ for all i, j, and ε such that $T_i^k(\varepsilon)$ and $T_j^k(\varepsilon)$ are well defined and $X + T_i^k(\varepsilon)$ and $X + T_j^k(\varepsilon) \in \Gamma$.

These axioms may be understood as follows. First, one can interpret $T_i^k(\varepsilon)$ as a taxation scheme. With this interpretation, $T_i^1(\varepsilon)$ consists in making individual i pay ε (without any transfer): one obtains $X + T_i^1(\varepsilon)$ (let $X \in \Gamma$, so $x_i \leqslant x_{i+1}$). Admittedly, a decision maker will consider that such a policy has a negative impact on welfare. The next question we can ask is: *Does the size of this negative impact depend on the location where this policy is applied*, i.e., the rank of the individual who has to pay ε? If not, the decision maker respects $(A1)$. But this implies that the decision maker is not concerned at all by equality: making the poorest one or the richest one paying ε is, from his point of view, equivalent. Now, assume that the decision maker thinks—which seems reasonable if he cares about inequality—that it is better to apply this policy to individual i rather than to individual $(i-1)$. We then have

$$(x_1, \ldots, x_{i-1}, x_i - \varepsilon, x_{i+1}, \ldots, x_n) \succcurlyeq (x_1, \ldots, x_{i-1} - \varepsilon, x_i, x_{i+1}, \ldots, x_n)$$

which for ε small enough (i.e., such that $x_{i-1}+\varepsilon \leqslant x_i-\varepsilon$) implies, by the independence axiom,

$$(x_1, ..., x_{i-1}+\varepsilon, x_i-\varepsilon, x_{i+1}, ..., x_n) \succcurlyeq (x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n).$$

That is: $X+T_i^2(\varepsilon) \succcurlyeq X$. In other words, it implies that the decision maker considers the policy $T_i^2(\varepsilon)$ (which is known as a Pigou–Dalton transfer) as a welfare improving one. He is inequality averse in the very standard sense: any Pigou–Dalton transfer increases his measure of welfare. But then, we can go further and ask whether the size of the increase of welfare implied by this policy depends on the location in the income distribution where this policy is applied. If the answer is no, the decision maker respects axiom ($A2$). Assume, on the other hand, that the location where this policy is applied is relevant. In particular, if the decision maker is quite sensitive to inequalities, it may be the case that the lower this policy is applied, the better it is—from the decision maker's point of view—(this is the "rank dependent" version of the principle of diminishing transfer: see e.g. Mehran [13], Kakwani [12], Chateauneuf and Wilthien [6], Zoli [17]). In this case we have

$$\begin{aligned}(x_1, ..., x_{i-2}+\varepsilon, x_{i-1}-\varepsilon, x_i, x_{i+1}, ..., x_n)&\\ \succcurlyeq (x_1, ..., x_{i-2}, x_{i-1}+\varepsilon, x_i-\varepsilon, x_{i+1}, ..., x_n)&\end{aligned}$$

which implies for ε small enough:

$$\begin{aligned}(x_1, ..., x_{i-2}-\varepsilon, x_{i-1}+2\varepsilon, x_i-\varepsilon, x_{i+1}, ..., x_n)&\\ \preccurlyeq (x_1, ..., x_{i-2}, x_{i-1}, x_i, x_{i+1}, ..., x_n).&\end{aligned}$$

That is $X+T_i^3(\varepsilon) \preccurlyeq X$. Therefore, the decision maker believes that the policy $T_i^3(\varepsilon)$ has a negative impact on welfare. Does the size of this impact depend on the location where this policy is applied? If not, the decision maker respects axiom ($A3$). If it does, then one can think that the lower this policy is applied, the higher is the cost in terms of welfare. Which in turn implies that the decision maker considers the policy $T_i^4(\varepsilon)$ as a favorable one, and so on.

Some remarks may be in order. First, note that $A(k-1)$ implies Ak. Second, for any $k \in \mathbb{N}$, axiom (Ak) is equivalent to the following one:

Axiom (Ak^*). $\forall X, Y \in \Gamma, X \succcurlyeq Y \Leftrightarrow X+T_i^k(\varepsilon) \succcurlyeq Y+T_j^k(\varepsilon)$ for all i, j, and ε such that $T_i^k(\varepsilon)$ and $T_j^k(\varepsilon)$ are well defined and $X+T_i^k(\varepsilon)$, $Y+T_j^k(\varepsilon)$, $X+T_j^k(\varepsilon)$, $Y+T_i^k(\varepsilon) \in \Gamma$.

Indeed, it is obvious that $(Ak)^*$ implies (Ak) (simply let $X=Y$ in (Ak^*)). Assume that (Ak) is satisfied. Then, for any X and Y in Γ and i, j, ε such that $X+T_i^k(\varepsilon)$, $X+T_j^k(\varepsilon)$, $Y+T_i^k(\varepsilon)$ and $Y+T_j^k(\varepsilon)$ belong to Γ,

$$X+T_i^k(\varepsilon) \sim X+T_j^k(\varepsilon)$$
$$Y+T_i^k(\varepsilon) \sim Y+T_j^k(\varepsilon).$$

If $X \succcurlyeq Y$, this implies $X+T_i^k(\varepsilon) \succcurlyeq Y+T_i^k(\varepsilon)$ by the independence axiom ($X, Y, X+T_i^k(\varepsilon)$ and $Y+T_i^k(\varepsilon)$ are clearly comonotone). Therefore, by transitivity, $X+T_i^k(\varepsilon) \succcurlyeq Y+T_j^k(\varepsilon)$.

Note that $(A2^*)$ is what Ben-Porath and Gilboa [1] called "Axiom $A6$ on F": It is the key axiom in their characterization of the Gini index.

We have the following result.

THEOREM 3.1. *For a decision maker who behaves in accordance with Yaari's dual model, with f $(k+1)$ times continuously differentiable on* $[0, 1]$, *the two following propositions are equivalent:*

(i) *f is a polynomial of degree (at most) k.*

(ii) *the decision maker respects axiom* (Ak).

Proof. See the appendix.

Now, we define the P-Gini indices as follows: a generalized Gini index IG is a P-Gini index if f is a polynomial. Theorems 1 and 2 imply readily the following corollary.

COROLLARY 3.1. *Assume that the decision maker behaves in accordance with Yaari's dual model, with f $(k+1)$ times continuously differentiable on* $[0, 1]$. *Then the two following propositions are equivalent:*

(i) *the decision maker respects axiom* (Ak).

(ii) *the decision maker only takes into account the welfare of coalitions with less than* $(k+1)$ *agents.*

Hence, we obtain a nice interpretation of P-Gini indices. If the decision maker respects axiom $(A1)$, we obtain an utilitarian welfare function, where only individuals' incomes are taken into account. If the decision maker respects $(A2)$ and not $(A1)$, he also considers the welfare of pairs: We then obtain the classical Gini index. As pointed out by Gilboa and Schmeidler [9], this may be interpreted as envy-aversion. Indeed, according to the canonical representation, the welfare of the pair $\{i, j\}$ is equal to $\min\{x_i, x_j\}$. Then, the coefficient α_2^f can be interpreted as the relative weight put by the decision maker on i's envy in j (assuming, say, that $x_i < x_j$). But, as noted by Gilboa and Schmeidler, if $\alpha_j^f = 0$ for all $j > 2$, the welfare function is, in one sense, linear in envy. Let us consider the example provided by these authors. Let $X=(0,1,1)$. If $\alpha_j^f=0$ for all $j>2$ (i.e., the decision maker respects $(A2)$), individual 1's envy in 2 and 3 is simply described by $\alpha_2^f[\min\{0,1\}]+\alpha_2^f[\min\{0,1\}]$. But, as Gilboa and Schmeidler put it: "More generally, one may suspect that individual 1 $(\cdots)$ will feel even "greater" envy that this "sum". After all, s/he may justly claim that "*Everyone* is better off than I am"." Now, assume that $\alpha_3^f>0$ (hence, the decision maker doesn't respect $A2$). Then, individual 1's envy in 2 and 3 is described by $\alpha_2^f[\min\{0,1\}]+\alpha_2^f[\min\{0,1\}]+\alpha_3^f[\min\{0,1,1\}]$. The welfare functional is no more linear in envy.

APPENDIX

We will need the following lemma.

LEMMA. *Let $f:[0,1]\mapsto[0,1]$ be $(k+1)$ times continuously differentiable on* $[0,1]$. *Then f is a polynomial of degree (at most) k if and only if:*

$$\exists t\in\mathbb{N},\quad \forall i,n\in\mathbb{N},\quad t\leqslant i\leqslant n,\quad n>0,\quad \Delta^{k+1}f\left(\frac{i}{n}\right)=0. \tag{1}$$

Proof. The "only if" part of the lemma is trivial. Let us prove the "if" part. Let $n=rm$ and $i=qm$, with r, q and m arbitrarily chosen in $\mathbb{N}$ such that $t\leqslant i\leqslant n$ and $n>0$. We have (see e.g. Cartan [4], Proposition 7.2.1, p. 92):

$$\left\|\Delta^{k+1}f\left(\frac{i}{n}\right)-\left(\frac{1}{n}\right)^{k+1}f^{(k+1)}\left(\frac{i}{n}\right)\right\|=o\left(\left(\frac{1}{n}\right)^{k+1}\right).$$

Therefore (1) implies:

$$\lim_{m\to\infty}(rm)^{k+1}\,\Delta^{k+1}\left(\frac{qm}{rm}\right)=\lim_{m\to\infty}(rm)^{k+1}\left(\frac{1}{rm}\right)^{k+1}f^{(k+1)}\left(\frac{qm}{rm}\right)$$
$$=f^{(k+1)}\left(\frac{i}{n}\right).$$

It follows that $f^{(k+1)}(p)=0$ for all $p\in[0,1]\cap\mathbb{Q}$; by density of $[0,1]\cap\mathbb{Q}$ in $[0,1]$ and continuity of $f^{(k+1)}$, this implies $f^{(k+1)}(x)=0$ for all $x\in[0,1]$, the desired result. ∎

Proof of Theorem 2.1.

(i) ⇒ (ii): If f is a polynomial with degree (at most) k, $\Delta^j f(\frac{i}{n})=0$ for any $j=k+1, k+2, \ldots$, and $i=j, \ldots, n$. So, in particular, $\Delta^j f(\frac{j}{n})=0$ for any $j=k+1, k+2, \ldots, n$. Therefore,

$$\alpha_j^f = \Delta^j f\left(\frac{j}{n}\right)=0, \qquad \forall j=k+1, k+2, \ldots, n$$

(ii) $\Rightarrow$ (i): Assume that for any $j>k$, $\alpha_j^f=0$. This implies that for any $j>k$, $\Delta^j f(\frac{j}{n})=0$. But we have, for fixed k, and for any $i \geqslant 0$, $l=0, \ldots, n$:

$$\begin{aligned}
\Delta^{k+i+1} f\left(\frac{l}{n}\right) &= (I-E)^i\,(I-E)^{k+1}\, f\left(\frac{l}{n}\right)\\
&= (I-E)^{k+1}\left[\sum_{j=0}^{i} (-1)^{i-j}\binom{i}{j} f\left(\frac{l-i+j}{n}\right)\right]\\
&= \sum_{j=0}^{i} (-1)^{i-j}\binom{i}{j} \Delta^{k+1} f\left(\frac{l-i+j}{n}\right).
\end{aligned}$$

So, in particular:

$$\Delta^{k+i+1} f\left(\frac{k+i+1}{n}\right)=\sum_{j=0}^{i} (-1)^{i-j}\binom{i}{j} \Delta^{k+1} f\left(\frac{k+1+j}{n}\right). \tag{2}$$

Let $\mathscr{P}(i)$: $[\Delta^{k+1} f(\frac{k+r}{n})=0, \forall r=1, \ldots, i]$. Since $\alpha_{k+1}^f=\Delta^{k+1} f(\frac{k+1}{n})=0$, $\mathscr{P}(1)$ is true. Assume that $\mathscr{P}(i)$ is true. By (2) we have:

$$\begin{aligned}
\Delta^{k+i+1} f\left(\frac{k+i+1}{n}\right) &= \sum_{j=0}^{i-1} (-1)^{i-j}\binom{i}{j} \Delta^{k+1} f\left(\frac{k+1+j}{n}\right)\\
&\quad + \Delta^{k+1} f\left(\frac{k+i+1}{n}\right).
\end{aligned}$$

But $\Delta^{k+i+1} f(\frac{k+i+1}{n})=\alpha_{k+i+1}^f=0$ by assumption, and $\sum_{j=0}^{i-1} (-1)^{i-j}\binom{i}{j}\times \Delta^{k+1} f(\frac{k+1+j}{n})=0$ by $\mathscr{P}(i)$. Therefore, $\Delta^{k+1} f(\frac{k+i+1}{n})=0$, hence $\mathscr{P}(i) \Rightarrow \mathscr{P}(i+1)$. This implies:

$$\Delta^{k+1} f\left(\frac{j}{n}\right)=0, \qquad \forall j, n \in \mathbb{N} \qquad \text{such that} \quad k<j \leqslant n$$

which in turn entails that f is a polynomial of degree (at most) equal to k by the lemma (obviously, the degree of f is exactly k if $\alpha_k^f \neq 0$). ∎

Proof of Theorem 3.1.

(i) $\Rightarrow$ (ii): Define $F: T_i^k(\varepsilon) \mapsto T_{i-1}^k(\varepsilon)$ and $J: T_i^k(\varepsilon) \mapsto T_i^k(\varepsilon)$. We have: $T_i^k(\varepsilon)=(J-F)^{k-1}\, T_i^1(\varepsilon)$. Therefore:

$$T_i^k(\varepsilon) = \sum_{l=0}^{k-1} (-1)^{k-1-l} \binom{k-1}{l} T_{i-k+1+l}^1(\varepsilon).$$

Let $j = k-1-l$. We then get:

$$T_i^k(\varepsilon) = \sum_{l=0}^{k-1} (-1)^j \binom{k-1}{k-1-j} T_{i-j}^1(\varepsilon) = \sum_{l=0}^{k-1} (-1)^j \binom{k-1}{j} T_{i-j}^1(\varepsilon)$$

since $\binom{k-1}{k-1-j} = \binom{k-1}{j}$.

The $(i-j)$ th component $T_i^k(\varepsilon)$ is $(-1)^j \binom{k-1}{j}(-\varepsilon)$ for $j = 0, \ldots, k-1$ and 0 otherwise. We hence have:

$$\begin{aligned} &V(X + T_i^k(\varepsilon)) - V(X) \\ &= \sum_{j=0}^{k-1} \left[f\left(\frac{n-i+j+1}{n}\right) - f\left(\frac{n-i+j}{n}\right) \right] (-1)^j \binom{k-1}{j} (-\varepsilon) \\ &= (-1)^k \varepsilon \sum_{j=0}^{k-1} \left[f\left(\frac{n-i+j+1}{n}\right) - f\left(\frac{n-i+j}{n}\right) \right] (-1)^{k-1-j} \binom{k-1}{j} \end{aligned}$$

(the last equality holds because $(2k-1-j)$ is even if and only if $(j+1)$ is even). Hence:

$$V(X + T_i^k(\varepsilon)) - V(X) = (-1)^k \varepsilon \Delta^k f\left(\frac{n-i+k}{n}\right)$$

Assume that f is a polynomial of degree k. Then $\Delta^k f(\frac{n-i+k}{n})$ is constant when i varies from k to n. Hence, $V(X + T_i^k(\varepsilon)) - V(X)$ doesn't depend on i. Therefore:

$$\forall X \in \Gamma, \quad V(X + T_i^k(\varepsilon)) - V(X) = V(X + T_j^k(\varepsilon)) - V(X), \quad \forall i, j \in \{k, \ldots, n\}$$

which implies (ii).

(ii) $\Rightarrow$ (i): Assume we have (ii):

$$\forall X \in \Gamma, \qquad X + T_i^k(\varepsilon) \sim X + T_j^k(\varepsilon)$$

for all i, j and ε such that $X + T_i^k(\varepsilon)$ and $X + T_j^k(\varepsilon)$ belong to Γ. Hence, $V(X + T_i^k(\varepsilon)) - V(X)$ doesn't depend on i, which implies:

$$(-1)^k \varepsilon \Delta^k f\left(\frac{n-i+k}{n}\right) = (-1)^k \varepsilon \Delta^k f\left(\frac{n-j+k}{n}\right)$$

$$\text{for all} \quad i \neq j \in \{k, \ldots, n\}$$

Therefore, $\Delta^{k+1} f(\frac{n-i+k}{n})=0$ for all i and n in $\mathbb{N}$ such that $n \geqslant i \geqslant k$, which implies that f is a polynomial of degree (at most) k by the lemma. (The degree of f is exactly k if the decision maker respects (Ak) but not $(A(k-1))$.) ▮

ACKNOWLEDGMENT

Financial support from the European Community (TMR Program) and the hospitality of Tel Aviv University, where part of this work was done, are gratefully acknowledged. I thank Eddie Dekel and Itzhak Gilboa for insightful discussions and comments.

REFERENCES

1. E. Ben-Porath and I. Gilboa, Linear measures, the Gini index and the income-equality trade-off, *J. Econ. Theory* **64** (1994), 443–467.
2. Deleted in proof.
3. W. Bossert, An axiomatization of the single-series Ginis, *J. Econ. Theory* **50** (1990), 82–92.
4. H. Cartan, "Calcul Differéntiel," Hermann, Paris, 1967.
5. S. Chakravarty, Extended Gini indices of inequality, *Int. Econ. Rev.* **29** (1988), 147–156.
6. A. Chateauneuf and P.-H. Wilthien, Third inverse stochastic dominance, Lorenz curves and favourable double transfer, Cahiers de la MSE, 1999.94, Université de Paris, 1999.
7. D. Donaldson and J. Weymark, A single-parameter generalization of the Gini indices of inequality, *J. Econ. Theory* **22** (1980), 67–86.
8. U. Ebert, Measurement of inequality: An attempt at unification and generalization, *Soc. Choice Welfare* **5** (1988), 147–169.
9. I. Gilboa and D. Schmeidler, Additive representations of non-additive measures and the Choquet integral, *Ann. Operation Res.* **52** (1994), 43–65.
10. I. Gilboa, and D. Schmeidler, Canonical representation of set functions, *Math. Operations Res.* **20** (1995), 197–212.
11. M. Grabisch, *K*-additive measures: Recent issues and challenges, *in* "5th Int. Conf. On Soft Computing and Information/Intelligent Systems, Iizuka, Japan, 1998."
12. N. C. Kakwani, On a class of poverty measures, *Econometrica* **48** (1980), 437–446.
13. F. Mehran, Linear measures of income inequality, *Econometrica* **44** (1976), 805–809.
14. J. Weymark, Generalized Gini inequality indices, *Math. Soc. Sci.* **1** (1981), 409–430.
15. M. Yaari, The dual theory of choice under risk, *Econometrica* **55** (1987), 95–115.
16. M. Yaari, A controversal proposal concerning inequality measurement, *J. Econ. Theory* **44** (1988), 381–397.
17. C. Zoli, Intersecting generalized Lorenz curves and the Gini index, *Soc. Choice Welfare* **16** (1999), 183–196.