

Decisions with conflicting and imprecise information

Thibault Gajdos · Jean-Christophe Vergnaud

Received: 9 February 2012 / Accepted: 5 July 2012
© Springer-Verlag 2012

Abstract When facing situations involving uncertainty, experts might provide imprecise and conflicting opinions. Recent experiments have shown that decision makers display aversion towards both disagreement among experts and imprecision of information. We provide an axiomatic foundation for a decision criterion that allows one to distinguish on a behavioral basis the decision maker's attitude towards imprecision and disagreement. This criterion accommodates patterns of preferences observed in experiments that are precluded by two-steps procedures, where information is first aggregated, and then used by the decision maker. This might be seen as an argument for having experts transmitting a more detailed information to the decision maker.

1 Introduction

When facing situations involving uncertainty, a decision maker might seek the advice of experts to obtain some information. This raises the following question: how to decide on the basis of information coming from several experts? Of course, different experts might have different opinions. Although the disagreement among them can be reduced through appropriate communication protocols and updating procedures, it is often the case that some divergences still persist. Moreover, experts could also

T. Gajdos (✉)
Aix-Marseille University (Aix-Marseille School of Economics), CNRS & EHESS, 2, rue de la Charité,
13236 Marseille Cedex 02, France
e-mail: thibault.gajdos@univ-amu.fr

J.-C. Vergnaud
CNRS, Centre d'Economie de la Sorbonne, 106-112 Boulevard de L'Hôpital,
75647 Paris Cedex 13, France
e-mail: vergnaud@univ-paris1.fr

provide imprecise information. We argue that one should take these two dimensions into account.

An extensive literature has been devoted to the use of experts' opinion in a probabilistic setting. Two approaches can be distinguished. On one hand, one may assume that the decision maker has a prior distribution, and use experts' assessments to update this prior. This leads to the bayesian theory initiated by [Morris \(1974, 1977\)](#).¹ On the other hand, there might be situations where the decision maker is unable (or doesn't want) to provide a prior distribution. This is the situation we consider in this paper. In such a case, the traditional approach consists in combining experts' opinions into a single piece of information, and then deciding on the basis of this aggregated information.² A famous example of such an aggregation procedure is the linear aggregation rule, that consists in computing a weighted average of experts' probability distributions.³

We argue that this route may not be appropriate if one wishes to take into account disagreement and imprecision.

Let us illustrate this difficulty with a stylized example. Suppose that experts are asked to give predictions about some events. We allow the experts to express their degrees of beliefs through probability intervals. The size of the intervals captures the imprecision of their opinions. We compare now three possible situations, described in the table below.

	Expert <i>a</i>	Expert <i>b</i>
Situation 1	$\frac{1}{2}$	$\frac{1}{2}$
Situation 2	$\frac{1}{4}$	$\frac{3}{4}$
Situation 3	$[\frac{1}{4}, \frac{3}{4}]$	$[\frac{1}{4}, \frac{3}{4}]$

Consider first situations 1 and 2. In both cases, experts provide precise assessments. According to the linear aggregation rule (and assuming that both experts are equally reliable), one would end in both situations with the same aggregated information, namely that the event will occur with probability $\frac{1}{2}$. However, these two situations are rather different, as in the first one experts agree, whereas in the second one they strongly disagree. The aggregation procedure fails to keep track of this disagreement.

A simple way out consists in aggregating experts opinions by probability intervals. A natural candidate in situation 2 would be the interval $[\frac{1}{4}, \frac{3}{4}]$. But consider now situation 3. Admittedly, any sensible aggregation rule should respect unanimity among experts. Thus the aggregation in situation 3 should also lead to the interval $[\frac{1}{4}, \frac{3}{4}]$. However, situations 2 and 3 greatly differ. Indeed, in situation 2, experts provide strongly conflicting but precise information, whereas in situation 3, they provide strongly imprecise but similar predictions. The aggregation procedure considered does not allow one to distinguish between imprecision of, and disagreement among, experts.

¹ See, e.g., [Clemen and Winkler \(2007\)](#) for a survey.

² See, e.g., [Cooke \(1991, chap. 11\)](#), for a survey.

³ This aggregating rule, applied to probability distributions, is known in the statistics literature as the "pooling rule". See [Stone \(1961\)](#), [McConway \(1981\)](#), and [Genest and Zidek \(1986\)](#) for a survey. It has been extended to lower probabilities by [Wagner \(1989\)](#). See among others [Nau \(2002\)](#) and [Troffaes \(2006\)](#) for aggregation rules for sets of probability distributions.

This example suggests that it might be tricky to find an aggregation procedure that takes into account simultaneously, in a satisfactory way, both imprecision of experts assessments and disagreement among them. This would not be a problem if decision makers were indifferent between ambiguity coming from imprecision of experts, or disagreement among them. The few experimental studies that have addressed this question to date suggest that such is not the case. In particular, [Smithson \(1999\)](#) introduces the distinction between “source conflict” and “source ambiguity”. He formulated the following “conflict aversion hypothesis”:

Likewise, conflicting messages from two equally believable sources may be more disturbing *in general* than two informatively equivalent, ambiguous, but agreeing messages from the same sources.

[Smithson \(1999, p. 184\)](#)

Smithson finds evidence supporting this hypothesis in experiments with students involving verbal statements. [Cabantous \(2007\)](#) and [Cabantous et al. \(2011\)](#) conducted experiments in a probabilistic setting (very similar to our example) with professional actuaries. They observe that insurance professionals tend to charge higher premiums under source conflict than under source ambiguity (which means in this context that situation 3 is preferred to situation 2), thereby providing further support for the conflict aversion hypothesis.

These results suggest that one should consider directly how a decision maker behaves when using information coming from several sources. However, to the best of our knowledge, there is no decision model that rationalizes the conflict aversion hypothesis. We axiomatically characterize preferences that exhibit independently aversion towards imprecision *and* disagreement. We obtain a two-step procedure. The first step consists in using separately experts assessments in a multiple prior model. The second step consists in an aggregation of these evaluations through a multiple weights model. Such a model is compatible with the evidence found by [Smithson \(1999\)](#); [Cabantous \(2007\)](#) and [Cabantous et al. \(2011\)](#). Moreover, we show that whenever this model reduces to a two-step procedure consisting in first aggregating opinions provided by experts, and then deciding on the basis of this aggregated information, it violates the “conflict aversion hypothesis”. This might be seen as an argument in favour of having experts commissions transmitting a more detailed information, including divergent opinions, to the decision maker.

The rest of the paper is organized as follows. We present the formal setup in Sect. 2. Section 3 is devoted to the axiomatic characterization of the decision maker preferences. We in particular introduce an axiom of disagreement aversion that can account for the conflict aversion hypothesis. In Sect. 4 we present results on imprecision aversion and disagreement aversion.

2 Setup

Probabilities have proved to be a powerful tool to summarize and transmit experts’ knowledge to policy makers, specially in situations of scientific uncertainty (see e.g. [Cooke 1991](#)). Therefore, the first step for making policy decisions in complex

situations (such as, for instance, climate changes) is to *elicit* experts beliefs. A large literature, going back to [Cooke \(1906\)](#) path-breaking contribution, has been devoted to the design of eliciting procedures for experts' probabilistic predictions. The Matching Probability rule is an example of such a procedure. It consists in offering the subject the possibility to exchange a lottery based on his subjective probability against a lottery with identical pay-off but known probability. Subjects are thus expected to accept such exchange as long as the objective probability is larger than their subjective one. The use of a Becker–DeGroot–Marschak mechanism guarantees that subjects reveal their true beliefs. The idea is that these predictions reflect their knowledge, the data they may observe, and the models or theories they use to interpret these data. Note that the fact that beliefs are expressed as probabilities is dictated by the elicitation procedure. It does not imply that experts do actually have precise beliefs.

In particular, assume that the data is scarce, or that the scientific knowledge is poor, as it is the case for instance concerning climatic changes. Then it might be too demanding to assume that experts' beliefs and predictions can take the form of precise probabilities. For instance, if the expert uses a theoretical model to build his predictions, but is uncertain about the exact values of the parameters of his model, he might provide the predictions obtained for a reasonable set of parameters. More generally, experts might provide sets of predictions (instead of precise predictions), reflecting either uncertainty in scientific knowledge (e.g., poor understanding of coupled physics phenomena, initiating events, fault trees or event trees), or imprecision (for instance due to error measurement) and scarceness (due to lack of observations) of data. Such an approach has been successfully used in climatic sciences by [Kriegler et al. \(2009\)](#), where experts express their beliefs on a given event through probability intervals. Accordingly, we will assume that experts' predictions can take the form of sets of probabilities.

It is important to note at this point that there is nothing irrational in facing two experts who provide distinct sets of predictions. It may simply reflect the fact that the two experts use different models, and it does not imply that there is some asymmetry of information. In particular, even after confronting their predictions, experts may well persist in their disagreement. In other words, in this framework, agreeing to disagree is not irrational. On the contrary, methods for combining experts predictions like the famous Delphi technique have precisely been criticized on the basis that they tend to “force” the consensus among experts, and thus lead to rather uninformative statements ([Rennie \(1981\)](#)). Finally, let us make clear that we assume that all relevant communication between experts has taken place and been taken into account before they submit their statements. The disagreement among experts' opinions, if any, is thus what remains when all communication, learning, and updating procedures have been implemented.

We now provide a formal definition of experts' prediction and decision maker's preferences. Let Ω be a finite set of states.⁴ Let $\Delta(\Omega)$ be the set of all probability distributions over Ω , and \mathcal{P} be the family of compact and convex subsets of $\Delta(\Omega)$, where compactness is defined with regard to the Euclidean space \mathbb{R}^Ω . The support of

⁴ The finiteness assumption is not needed, and is only made for sake of simplicity. All our results extend to the denumerable case.

$P \in \mathcal{P}$, denoted $supp(P)$, is defined as the union over $p \in P$ of the support of p . Finally, for all $s, t \in \Omega$, let δ_s be the probability distribution defined by $\delta_s(s) = 1$, and $\Delta_{st} = co(\{\delta_s, \delta_t\})$.⁵

The collection of sets \mathcal{P} is a mixture space under the operation defined by

$$\lambda P + (1 - \lambda)Q = \{\lambda p + (1 - \lambda)q : p \in P, q \in Q\}.$$

The set of pure outcomes is denoted by X . Let $\Delta(X)$ be the set of simple lotteries (probability measures with finite supports) over X . Let $\mathcal{F} = \{f : \Omega \rightarrow \Delta(X)\}$ be the set of acts. Abusing notation, any lottery is viewed as a constant act which delivers that lottery regardless of the states. The set \mathcal{F} is a mixture space under the operation defined by:

$$(\alpha f + (1 - \alpha)g)(\omega) = \alpha f(\omega) + (1 - \alpha)g(\omega), \forall \omega \in \Omega.$$

For $E \subseteq \Omega$, denote by $f_E g$ the act that yields $f(\omega)$ if $\omega \in E$ and $g(\omega)$ if not.

The decision maker is endowed with a preference relation \succsim defined on $\mathcal{F} \times \mathcal{P} \times \mathcal{P}$. When

$$(f, P_1, P_2) \succsim (g, Q_1, Q_2),$$

the decision maker prefers choosing f when expert i 's prediction is P_i ($i \in \{1, 2\}$) to choosing g when expert i 's prediction is Q_i ($i \in \{1, 2\}$). Note that the fact that the decision maker has preferences over triplets (f, P_1, P_2) is not standard, in the sense that it implies that she can compare the same act under different informational settings. This is meaningful insofar as we do not postulate, as does [Savage \(1954\)](#), that the state space represents the set of the worlds. It is for us a mere coding device, without any substantial existence. For instance, coming back to our introductory example, the problem of betting on an event in situation 1, 2 or 3 can be written as a problem of choosing between $(f, \{\frac{1}{2}\delta_s + \frac{1}{2}\delta_t\}, \{\frac{1}{2}\delta_s + \frac{1}{2}\delta_t\})$, $(f, \{\frac{1}{4}\delta_s + \frac{3}{4}\delta_t\}, \{\frac{3}{4}\delta_s + \frac{1}{4}\delta_t\})$ and $(f, co(\{\frac{1}{4}\delta_s + \frac{3}{4}\delta_t, \frac{3}{4}\delta_s + \frac{1}{4}\delta_t\}), co(\{\frac{1}{4}\delta_s + \frac{3}{4}\delta_t, \frac{3}{4}\delta_s + \frac{1}{4}\delta_t\}))$, where f is a bet on state s .

A distinctive feature of our framework is thus that the set of experts is fixed, and that they provide various predictions. As an example, consider the case of specialized experts panels, who are solicited on a regular basis for a large number of problems (for instance bioethical committees, giving recommendations concerning genetically modified organisms). The relation between the decision maker and the experts is repeated, and therefore it makes sense to consider such preference relation with varying predictions for the two experts. Moreover, we will assume that the experts have already been selected and evaluated, and that the decision maker cannot deduce anything more regarding their reliability from their statements. With this two assumptions (fixed set of experts and given reliability) we disregard the important problem of the design and selection of experts committees. We leave these questions for future research. We focus here on the last step of the decision process, namely the decision itself.

⁵ Given two sets P and Q , $co(P, Q)$ denotes the convex hull of P and Q .

To summarize, we make here the following implicit assumptions:

- The information (P_1, P_2) is what is available after all discussions among experts;
- The committee size and composition is fixed;
- The decision maker cannot deduce anything concerning experts reliability from their statements.

3 Representation

The celebrated Maxmin Expected Utility, suggested by [Gilboa and Schmeidler \(1989\)](#), is now largely accepted as the simplest and most prominent model of decision with multiple priors. It states that an act f is evaluated by

$$\min_{p \in C} E_p u(f),$$

where $E_p u(f)$ denotes the expected utility of f with respect to probability p and utility u , and C is a set of probabilities. Thus, the decision maker behaves as if she evaluates act f by its worst possible expected utility over C . However, C does not necessarily represents the information available to the decision maker. For instance, she might believe that any probability distribution is possible, but nevertheless behaves as if only one was relevant. In that case, C reduces to a singleton, and the decision maker behaves as a Bayesian. On the other hand, in a similar informational context, the decision maker may be extremely cautious, in which case C would be the set of all possible probabilities. Thus C is related to the information available to the decision maker and her attitude towards uncertainty. As [Gilboa and Schmeidler \(1989\)](#) focused on decisions made in a given informational context, they left implicit how the set C is related to the information available to the decision maker.

[Gajdos et al. \(2008a\)](#) clarify the link between the set C and the information available to the decision maker, when this information is an objective set of probabilities. In order to do so, they consider an extended framework, and assume that the decision maker has preferences over couples of acts ($f \in \mathcal{F}$) and information ($P \in \mathcal{P}$). In this setting, and act f with an information P is evaluated by

$$\min_{p \in \Gamma(P)} E_p u(f),$$

where $\Gamma : \mathcal{P} \rightarrow \mathcal{P}$ describes how the set of probabilities used to evaluate an act is related to the available information. In particular, a Bayesian decision maker is characterized by the fact that $\Gamma(P)$ is a singleton for all possible P , whereas a decision maker who is extremely cautious would be characterized by $\Gamma(P) = P$.

As a starting point, we simply assume here that Gilboa–Schmeidler Maxmin Expected Utility model still holds when the decision maker faces *two* sets of probabilities. In that case, it is natural to assume that the set of probabilities used to evaluate an act is related to the two pieces of available information. In other words, assuming that the first expert provides information P and the second expert provides information Q , an act f would be evaluated by

$$\min_{p \in \psi(P, Q)} E_p u(f).$$

We will also require the following extension of von-Neumann–Morgenstern’s independence axiom to hold. Assume that $(f, P, P) \succ (f, Q, Q)$. This means that the decision maker prefers f when both experts predict P to f when both experts predict Q . Assume that, in both situations, experts now consider that the true probability distribution could belong to a same set R with some probability α . They thus now predict $\alpha R + (1 - \alpha)P$ if they first predicted P , and $\alpha R + (1 - \alpha)Q$ if they first predicted Q . Applying the logic of von-Neumann–Morgenstern’s independence axiom probability-wise yield that the decision maker’s ranking should not be affected by this change, and thus that $(f, \alpha R + (1 - \alpha)P, \alpha R + (1 - \alpha)P) \succ (f, \alpha R + (1 - \alpha)Q, \alpha R + (1 - \alpha)Q)$. This requirement is no more (but no less) compelling than the usual independence axiom for lotteries. It implies that, for all $P, Q \in \mathcal{P}$, and all $\alpha \in (0, 1)$, $\psi(\alpha P + (1 - \alpha)Q, \alpha P + (1 - \alpha)Q) = \alpha\psi(P) + (1 - \alpha)\psi(Q)$ (see [Gajdos et al. \(2008a\)](#)). In other words, ψ is linear on the set $\{(P, P) : P \in \mathcal{P}\}$ of unanimous assessments.

Finally, although not indisputable, it makes sense to assume that the decision maker should not consider probabilities that are excluded by the two experts and cannot be obtained as a convex combination of some of the experts predictions. This means that $\psi(P, Q) \subseteq \text{co}(P \cup Q)$. As we will see, most of our results can be readily adapted if one relaxes this condition. We mainly keep it because we find it reasonable. These assumptions are formally stated below.

Axiom 1 (*Maxmin preferences*) There exist a function $V : \mathcal{F} \times \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$ which represents \succ , a mixture-linear function $u : \Delta(X) \rightarrow \mathbb{R}$, a mapping $\psi : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ linear on $\{(P, P) : P \in \mathcal{P}\}$ and satisfying $\psi(P, Q) \subseteq \text{co}(P \cup Q)$ such that:

$$V(f, P, Q) = \min_{p \in \psi(P, Q)} E_p u(f).$$

Moreover u is unique up to a positive linear transformation, and ψ is unique.

The next axiom states that if (i) the decision maker prefers f when both experts agree on an information set P_1 to g when both experts agree on P_2 , and (ii) she also prefers f when both experts agree on an information set Q_1 to g when both experts agree on Q_2 , then she prefers f when experts’ opinions are (P_1, Q_1) to g when experts’ opinions are (P_2, Q_2) . This is essentially a dominance axiom, similar in substance to the traditional Pareto requirement.

Axiom 2 (*Dominance*) For all $f, g \in \mathcal{F}, P_1, Q_1, P_2, Q_2 \in \mathcal{P}$,

$$\left. \begin{aligned} (f, P_1, P_1) \succ (g, P_2, P_2) \\ (f, Q_1, Q_1) \succ (g, Q_2, Q_2) \end{aligned} \right\} \Rightarrow (f, P_1, Q_1) \succ (g, P_2, Q_2)$$

Moreover, if one of the preferences on the left-hand side is strict, so is the preference on the right-hand side.

This axiom implies that information matters only insofar as it has an impact on the valuation of acts. Consider for instance, the case where $(f, P, P) \sim (f, Q, Q)$.

Axiom 2 implies that $(f, P, Q) \sim (f, P, P)$, even if $P \neq Q$. A simple aggregating rule that does not satisfy this Axiom is the following. If $P \cap Q \neq \emptyset$, then use $P \cap Q$ as an aggregated information; otherwise, use $\text{co}(P \cup Q)$. The intuition behind this rule is simple. If $P \cap Q \neq \emptyset$, the decision maker is confident enough to forget the disagreement among experts. On the other hand, if $P \cap Q = \emptyset$ she would be much more cautious and consider as plausible all scenario provided by both experts (as well as their convex combinations). To see why this rule is incompatible with Axiom 2, consider the following example. Assume there are two states, s_1 and s_2 , and let f be the act “getting 1 if s_1 occurs, and 0 otherwise”. Without loss of generality, assume that the utility of getting 1 is equal to 1 and that the utility of getting 0 is equal to 0. Let $P_1 = \{(p, 1-p) | p \geq \frac{1}{4}\}$, $P_2 = \{(p, 1-p) | \frac{1}{2} \geq p \geq \frac{1}{4}\}$, $Q_1 = Q_2 = \{(p, 1-p) | \geq \frac{3}{4}\}$. Observe that $P_1 \cap Q_1 = Q_1$, $P_2 \cap Q_2 = \emptyset$ and $\text{conv}(P_2 \cup Q_2) = P_1$. Assume that $V(g, P, P) = \min_{p \in P} E_p u(g)$, so that $(f, P_1, P_1) \sim (f, P_2, P_2)$. By Axiom 2 this implies that $(f, P_1, Q_1) \sim (f, P_2, Q_2)$, and given the aggregation rule under scrutiny we should have $(f, Q_1, Q_1) \sim (f, P_1, P_1)$. This implies in turn $\min_{p \in P_1} E_p u(f) = \frac{1}{4} = \min_{p \in Q_1} E_p u(f) = \frac{3}{4}$, a contradiction. More generally, Axiom 2 excludes the possibility that the decision maker infers the degree of reliability of the experts from the degree of disagreement among them.

This is consistent with assumptions made at the end of Sect. 2. Namely, we assume that the pool of experts has been selected and evaluated before they submit their reports, and that the decision maker’s confidence into them is not affected by their reports. This would in particular be the case when the pool of experts is a stable committee chosen to address a large number of issues. Of course, the problem of choosing the number of experts and selecting them is important, but it requires an extended and dynamic framework, in which preferences are defined on triplets involving acts, experts’ opinions, and committees. Put differently, as a first step, our setup is essentially static, and ignores the question of experts selection.

It is worth exploring the consequences of Axiom 2 in the traditional case, where each expert provides a single probability distribution, and the decision maker is an expected utility maximizer. Assume there exist a von Neumann–Morgenstern function u and an aggregation rule $\varphi : \Delta(\Omega) \times \Delta(\Omega) \rightarrow \Delta(\Omega)$.⁶ Axiom 2, together with a weak unanimity condition, then implies that the aggregated probability of an event only depends on the probabilities assigned to that event by the experts. It has been shown by McConway (1981) that this property, called the *Strong Setwise Function Property* characterizes the linear aggregation rule, as soon as there are at least three different states in Ω . We thus have the following result.

Proposition 1 *Let $|\Omega| \geq 3$. Assume there exists a von Neumann–Morgenstern function u and an aggregation rule $\varphi : \Delta(\Omega) \times \Delta(\Omega) \rightarrow \Delta(\Omega)$. such that for all $f, g \in \mathcal{F}$, $p_1, p_2, q_1, q_2 \in \Delta(\Omega)$,*

$$(f, p_1, q_1) \succ (g, p_2, q_2) \Leftrightarrow E_{\varphi(p_1, q_1)} u(f) \geq E_{\varphi(p_2, q_2)} u(f).$$

⁶ Note that these assumptions are weaker than those implied by 1, since we do not impose linearity and convexity conditions on φ .

Then, the following statements are equivalent:

- (i) Axiom 2 holds and $\varphi(p, p) = p$ for all $p \in \Delta(\Omega)$;
- (ii) There exists $\alpha \in (0, 1)$ such that for all $p, q \in \Delta(\Omega)$, $\varphi(p, q) = \alpha p + (1 - \alpha)q$.

This result makes clear that Axiom 2 strongly constrains the aggregation procedure. It is also clear that it precludes to take into account some aspects of the possible conflicts among experts. The aim of our last axiom is precisely to describe what differences among experts may actually affect the decision maker.

Given two information sets P, Q and $\alpha \in (0, 1)$, the information set $\alpha P + (1 - \alpha)Q$ can be seen as a compromise between P and Q in the following sense. It is the set of probability distributions obtained if one considers that the true probability distribution belongs to P with probability α , whereas it belongs to Q with probability $(1 - \alpha)$. Our axiom says that the decision maker always prefers when experts come in with opinions that are less disparate, i.e., always prefers $(f, \alpha P + (1 - \alpha)Q, \alpha Q + (1 - \alpha)P)$ to (f, P, Q) .

Axiom 3 (Disagreement aversion) For all $f \in \mathcal{F}, P, Q \in \mathcal{P}, \alpha \in (0, 1)$,

$$(f, \alpha P + (1 - \alpha)Q, \alpha Q + (1 - \alpha)P) \succcurlyeq (f, P, Q).$$

3.1 Main result

We derive from our axioms the following representation.

Theorem 1 Axioms 1, 2 and 3 are satisfied iff there exist a mixture-linear function $u : \Delta(X) \rightarrow \mathbb{R}$, a linear mapping $\varphi : \mathcal{P} \rightarrow \mathcal{P}$ satisfying $\varphi(P) \subseteq P$ and a symmetric closed and convex subset Π in $\Delta(\{1, 2\})$ such that \succcurlyeq can be represented by:

$$V(f, P, Q) = \min_{\pi \in \Pi} \left[\pi(1) \left(\min_{p \in \varphi(P)} \sum_{\omega} u(f(\omega))p(\omega) \right) + \pi(2) \left(\min_{p \in \varphi(Q)} \sum_{\omega} u(f(\omega))p(\omega) \right) \right].$$

Moreover u is unique up to a positive linear transformation, and φ and Π are unique.

First, note that if one relaxes the condition $\psi(P, Q) \subseteq \text{co}(P \cup Q)$ in Axiom 1, the only change in Theorem 1 would be that the $\varphi(P)$ should not be constrained to belong to P . Observe also that, if $\varphi(P) = P$, and $\Pi = \Delta\{1, 2\}$, then $V(f, P, Q) = \min_{p \in \text{co}(P \cup Q)} \sum_{\omega} u(f(\omega))p(\omega)$.

Maximizing this formula can be thought of as a two-step procedure. First, the decision maker transforms experts information through φ , and uses the resulting sets of probability to evaluate the act under consideration. Second, she aggregates linearly these two evaluations, using the worst weight vector in a set Π . The first step deals with experts' assessments. It is important to observe that we face two very different kinds of sets of probability distributions. Indeed, P and Q are strictly informational. They capture the information available to the decision maker. $\varphi(P)$ and $\varphi(Q)$ are the behavioural beliefs the decision maker would use to evaluate acts if she were facing

either P or Q .⁷ The mapping φ introduces a subjective treatment of an imprecise information. In the second step, evaluations based on information provided by experts are aggregated. The set Π captures the decision maker's attitude towards disagreement among valuations based on experts assessments.

Note that we also have:

$$V(f, P, Q) = \min_{p \in \Pi \otimes (\varphi(P), \varphi(Q))} \sum_{\omega} u(f(\omega)) p(\omega),$$

where $\Pi \otimes (\varphi(P), \varphi(Q)) = \{\pi(1)p + \pi(2)q \mid \pi \in \Pi, p \in \varphi(P), q \in \varphi(Q)\}$. Thus maximizing this formula can be thought of as first aggregating experts information through $\Pi \otimes (\varphi(P), \varphi(Q))$, and then applying the maxmin expected utility criterion over this set. Therefore the set $\Pi \otimes (\varphi(P), \varphi(Q))$, which corresponds to the set $\psi(P, Q)$ in Axiom 1, and has only a behavioral meaning: it is related to the decision maker's preferences.

Finally, note that a different route is also possible: It consists in aggregating *behavioral* beliefs of experts into behavioral beliefs of the decision maker. It essentially amounts to ask the experts what their own decision would be (possibly assuming that they would evaluate consequences the same way as the decision maker) and to aggregate these stated preferences. A large literature in social choice theory has been elaborated along these lines, following Harsanyi (1955) seminal paper.⁸ See among others (Mongin 1995; Gilboa et al. 2004; Gajdos et al. 2008b; Chambers and Echenique 2009; Nascimento 2012).

Indeed, this problem has also been addressed by Crés et al. (2011), who consider the problem of aggregating preferences of Maxmin Expected Utility maximizers who share the same utility function into a Maxmin Expected utility. They show that under Pareto constraint, the aggregate set of priors takes the form $\Pi \otimes (P_1, \dots, P_n)$, where P_i denotes individual i 's set of priors.⁹ They thus aggregate behavioral beliefs into behavioral beliefs.¹⁰ Finally, Nascimento (2012) considers the problem of aggregating preferences of experts who agree on the ranking of risky prospects, but have different perception of or attitude towards ambiguity. He obtains an aggregation rule that, when restricted to maxmin preferences, is a generalization of the rule of Crés et al. (2011) (although in a different framework and with a different justification).

4 Attitude towards information

4.1 Uncertainty and disagreement

We now turn to the behavioral characterization of imprecision and disagreement aversion. We use the standard comparative approach. Note that all the results in

⁷ In Gilboa and Schmeidler (1989) celebrated maxmin expected utility model, only behavioural beliefs appear.

⁸ Although Harsanyi (1955) actually considers the case where experts share the same beliefs, and disagree on valuations. But it really laid the foundations for the aggregation of experts' behavioral beliefs.

⁹ The two papers were independently developed.

¹⁰ Actually, one can also interpret experts' beliefs in Crés et al. (2011) as informational beliefs. But then it implies that the decision maker is forced to be extremely averse towards uncertainty.

this subsection remain true if one relaxes the condition $\psi(P, Q) \subseteq \text{co}(P \cup Q)$ in Axiom 1.

We first define comparative imprecision aversion as in Gajdos et al. (2004) and Gajdos et al. (2008a). Let a and b be two decision makers. We will say that b is more averse to imprecision than a if, whenever a prefers a precise situation to an imprecise one, so does b . In order to control for risk aversion, this definition is restricted to acts whose consequences are lotteries over two outcomes only (binary acts). Formally, for all $(\bar{x}, \underline{x}) \in X^2$, we define a corresponding set of binary acts as $\mathcal{F}_{\bar{x}, \underline{x}}^b = \{f \in \mathcal{F} \mid \exists p_s \in [0, 1], \text{ s.t. } f(s) = (\bar{x}, p_s; \underline{x}, 1 - p_s), \forall s \in \Omega\}$, where $(\bar{x}, p_s; \underline{x}, 1 - p_s)$ denotes the lottery that yields \bar{x} with probability p_s and \underline{x} with probability $(1 - p_s)$.

Definition 1 Let \succsim_a and \succsim_b be two preference relations defined on $\mathcal{F} \times \mathcal{P} \times \mathcal{P}$. Suppose there exist two prizes \bar{x}, \underline{x} in X such that both a and b strictly prefer \bar{x} to \underline{x} . We say that \succsim_b is more averse to imprecision than \succsim_a whenever for all $f \in \mathcal{F}_{\bar{x}, \underline{x}}^b, p \in \Delta(\Omega), P \in \mathcal{P}$

$$(f, \{p\}, \{p\}) \succsim_a (f, P, P) \Rightarrow (f, \{p\}, \{p\}) \succsim_b (f, P, P).$$

The following proposition shows how φ is related to the decision maker’s attitude towards imprecision. Intuitively, the more φ “shrinks” P , the less imprecision averse is the decision maker.

Proposition 2 *The following assertions are equivalent:*

1. \succsim_b is more averse to imprecision than \succsim_a .
2. for all $P \in \mathcal{P}, \varphi_a(P) \subset \varphi_b(P)$.

The following example illustrates this result in the simple case where there are only two states.

Example 1 Let $\Omega = \{s, t\}, \varphi(P) = \{(1 - \theta)c(P) + \theta p \mid p \in P\}$, where $\theta \in [0, 1]$ and $c(P)$ is the center of P , and $\Pi = \{(1 - \alpha)(\frac{1}{2}, \frac{1}{2}) + \alpha(t, 1 - t) \mid t \in [0, 1]\}$, where $\alpha \in [0, 1]$. In other words, $\varphi(P)$ is a contraction of P around its center, with a contraction rate equal to $(1 - \theta)$, whereas Π is a symmetric set of probabilities on $\Delta(\{s, t\})$.

In view of Proposition 2, θ can be interpreted as a measure of imprecision aversion (imprecision aversion increases with θ).

Comparative disagreement aversion will be defined along the same line as comparative imprecision aversion. It simply states that decision maker b is more averse to disagreement than decision maker a if, whenever a prefers a consensual situation to a situation with divergent information, so does b .

Assume that a decision maker strictly prefers (f, P, P) to (f, Q, Q) . Then (by Axiom 2) she will have the following preferences: $(f, P, P) \succ (f, P, Q) \succ (f, Q, Q)$. Now, consider $R(\alpha) = \alpha P + (1 - \alpha)Q$. Of course, the larger α , the better $(f, R(\alpha), R(\alpha))$. Since $(f, R(1), R(1)) \succ (f, P, Q) \succ (f, R(0), R(0))$, there is an (unique) $\hat{\alpha}$ such that $(f, R(\hat{\alpha}), R(\hat{\alpha})) \sim (f, P, Q)$. Loosely speaking, $R(\hat{\alpha})$ is

the worst *consensual* information that the decision maker considers as equivalent to (P, Q) when facing act f . Thus $(1 - \hat{\alpha})$ can be seen as the “price” she is ready to pay to avoid disagreement. Decision maker b is more averse to disagreement than decision maker a if b is ready to “pay” a higher price to avoid disagreement.

Definition 2 Let \succsim_a and \succsim_b be two preference relations defined on $\mathcal{F} \times \mathcal{P} \times \mathcal{P}$. We say that \succsim_b is *more averse to disagreement than* \succsim_a if for all $f \in \mathcal{F}$, $P, Q \in \mathcal{P}$, $\alpha \in (0, 1)$, such that both a and b prefers (f, P) to (f, Q) if :

$$(f, \alpha P + (1 - \alpha)Q, \alpha P + (1 - \alpha)Q) \succsim_a (f, P, Q)$$

then,

$$(f, \alpha P + (1 - \alpha)Q, \alpha P + (1 - \alpha)Q) \succsim_b (f, P, Q).$$

The following Proposition shows how Π is related to the decision maker’s attitude towards disagreement. Intuitively, the larger Π , the more averse to disagreement is the decision maker.

Proposition 3 *The following assertions are equivalent:*

1. \succsim_b is more averse to disagreement than \succsim_a ,
2. $\Pi_a \subseteq \Pi_b$.

Example 2 Let φ and Π as in Example 1. In view of Proposition 3, α can be interpreted as a measure of disagreement aversion (disagreement aversion increases with α).

4.2 Conflict aversion hypothesis and reduction

We now turn to the “conflict aversion hypothesis” postulated by [Smithson \(1999\)](#), and experimentally observed by [Cabantous \(2007\)](#) and [Cabantous et al. \(2011\)](#). According to this hypothesis, one typically observes that decision makers prefers situations where uncertainty is due to the vagueness of experts to situations where it comes from the disagreement among sharp experts. For instance, in the two states case, decision makers prefer a situation where experts beliefs on state s are $[\underline{p}, \bar{p}]$ to a situation where expert 1 believes that state s will occur with probability \underline{p} , whereas expert 2 believes that it will occur with probability \bar{p} . In order to study the conflict aversion hypothesis in our model, we first need to translate it formally, which is done by the following axiom.

Axiom 4 (*Conflict Aversion Hypothesis*) For all $\{p_1\}, \{p_2\} \in \mathcal{P}$, if $p_1 \neq p_2$, then:

$$\begin{cases} \forall f \in \mathcal{F}, (f, \text{co}\{p_1, p_2\}, \text{co}\{p_1, p_2\}) \succ (f, \{p_1\}, \{p_2\}) \\ \exists g \in \mathcal{F}, (g, \text{co}\{p_1, p_2\}, \text{co}\{p_1, p_2\}) \succ (g, \{p_1\}, \{p_2\}) \end{cases}$$

The traditional aggregation approach assumes that decisions with information coming from various experts can be made in two steps. First, information provided by

experts is somehow aggregated into a unique piece of information; then this aggregated information is used by the decision maker, who transforms it into behavioral beliefs. The problem can then be reduced to independent questions. First, how should we aggregate experts opinions? Second, what should the decision be, given this aggregated information? It is natural to wonder whether it is always possible to reduce the behavioral approach we follow to the traditional two-steps aggregation. This would be the case if any pair of statements is equivalent for the decision maker (from the informational point of view) to some congruent statements from the two experts. In other words, for all P, Q , there should exist R such that for all f , $(f, P, Q) \sim (f, R, R)$. We formalize this idea with the following Reduction Axiom.

Axiom 5 (Reduction) For all $P_1, P_2 \in \mathcal{P}$, there exists $R \in \mathcal{P}$ such that $\text{supp}(R) \subseteq \text{supp}(P_1 \cup P_2)$ and, for all $f \in \mathcal{F}$, $(f, P_1, P_2) \sim (f, R, R)$.

In order to provide a clear answer to that question we must get rid of some pathological situations. In particular, we must avoid the rather strange case where a more precise information yields to a worse evaluation for all acts (which would translate into the fact that, for some $P, Q \in \mathcal{P}$ such that $P \subsetneq Q, \varphi(Q) \subsetneq \varphi(P)$). This is done by the following axiom which stipulates that, if there is no conflict among experts, an increase in the precision of the information they deliver translates into a greater evaluation of at least one act.

Axiom 6 (Preference for Precision of Information) For all $P, Q \in \mathcal{P}$, if $P \subsetneq Q$ then either $(f, P, P) \sim (f, Q, Q)$ for all $f \in \mathcal{F}$, or there exists $g \in \mathcal{F}$ such that $(g, P, P) \succ (g, Q, Q)$.

The following Proposition shows that, under this rather innocuous axiom, the Reduction Axiom and the Conflict Aversion Hypothesis are incompatible. In other words, within our framework, our procedure can accommodate patterns of preferences that cannot be explained by a standard two-step procedure.

Proposition 4 *If (Preference for Precision of Information) and (Reduction) hold, then (Conflict Aversion Hypothesis) is not satisfied.*

The following example illustrates this result in the particular case of the preferences defined in Example 1. In this case, (Reduction) holds if, and only if, the Conflict Aversion Hypothesis is violated. On the other hand, the model can also accommodate the Conflict Aversion Hypothesis for a wide range of the parameters.

Example 3 Assume $|\Omega| = 2$ and let $\varphi(P)$ and Π be as in Example 1. Then the following statements are equivalent:

1. Reduction holds.
2. $\theta \geq \alpha$.
3. Conflict aversion hypothesis does not hold.

Finally, we suggested in the introduction that a natural way to aggregate information so as to take into account the decision maker attitude towards disagreement consists in considering the unions of the pieces of information provided by the experts. It is thus a natural question to ask if this rule can be obtained in our framework. The answer is: yes. It actually corresponds to the case of extreme aversion towards both disagreement and imprecision.

Proposition 5 *The two statements are equivalent:*

- (i) For all $P, Q \in \mathcal{P}$, $(f, P, Q) \sim (f, \text{co}(P \cup Q), \text{co}(P \cup Q))$
- (ii) $\Pi = \Delta_{12}$ and $\forall P \in \mathcal{P}, \varphi(P) = P$.

Acknowledgements We thank I. Gilboa, Ch. Gollier, B. Lipman, N. Vieille, Peter Wakker, and audience at the Toulouse School of Economics Theory seminar for useful comments and suggestions. We thank an anonymous referee for helpful comments. Financial support from ANR ComSoc (ANR-09-BLAN-0305-03) is gratefully acknowledged.

Appendix: Proofs

A.1 Proof of Proposition 1

We assume in the sequel that $(f, p_1, q_1) \succ (g, p_2, q_2)$ iff $E_{\varphi(p_1, q_1)}u(f) \geq E_{\varphi(p_2, q_2)}u(g)$. Let $U(f, p, q) = E_{\varphi(p_1, q_1)}u(f)$. We first need to state formally in our setup the *Strong Setwise Function Property* proposed by [McConway \(1981\)](#).

Strong Setwise Function Property (SSFP). There exists a function $\tilde{\varphi} : [0, 1]^2 \rightarrow [0, 1]$ such that for all $p, q \in \Delta(\Omega)$ and all $\sigma \in \Omega$, $\varphi(p, q)(\sigma) = \tilde{\varphi}(p(\sigma), q(\sigma))$.

We also need the following Unanimity condition.

Unanimity. For all $p \in \Delta(\Omega)$, $\varphi(p, p) = p$.

Now, assume there exists σ, p_1, q_1, p_2 and q_2 such that $p_1(\sigma) = p_2(\sigma), q_1(\sigma) = q_2(\sigma)$ and $\varphi(p_1, q_1)(\sigma) \neq \varphi(p_2, q_2)(\sigma)$. Define f such that $u(f(s)) = 0$ for all $s \neq \sigma$ and $u(f(\sigma)) = 1$. By (Unanimity), $U(f, p_1, p_1) = \sum_s p_1(s)u(f(s))$ and $U(f, p_2, p_2) = \sum_s p_2(s)u(f(s))$. Therefore, $U(f, p_1, p_1) = U(f, p_2, p_2)$. Similarly, $U(f, q_1, q_1) = \sum_s q_1(s)u(f(s)) = \sum_s q_2(s)u(f(s)) = U(f, q_2, q_2)$. Thus by dominance $U(f, p_1, q_1) = U(f, p_2, q_2)$, and therefore $\varphi(p_1, q_1)(\sigma) = \varphi(p_2, q_2)(\sigma)$, a contradiction. Thus SSFP is satisfied.

By Theorem 3.3 in [McConway \(1981\)](#), we know that, whenever $|\Omega| \geq 3$, SSFP is satisfied iff φ is the linear pooling rule. By Axiom 2, it must moreover be the case that no expert receive zero weight. Conversely, it is straightforward to check that the linear pooling rule with positive weights satisfies (Unanimity) and Axiom 2.

A.2 Proof of Theorem 1

Necessity is easily checked. We thus only prove sufficiency.

Let $U(f, P) = V(f, P, P)$ for all $f \in \mathcal{F}$ and $P \in \mathbb{P}$, where V is defined by Axiom 1, and $\mathcal{U} = \text{range } U$. Since u in Axiom 1 is unique up to a positive linear transformation, we can choose it such that $U(h_1, P) = 1$ and $U(h_2, P) = -1$ for some constant acts h_1, h_2 , and for any P (by unicity of ψ in Axiom 1, \succ is not degenerated, and thus h_1, h_2 exist). Note that \mathcal{U} is convex. Indeed, let $f, g \in \mathcal{F}$ and $P, Q \in \mathcal{P}$. By linearity of u and convexity of $\Delta(X)$, there exist constant acts \bar{f} and \bar{g} such that $U(f, P) = U(\bar{f}, P)$ and $U(g, Q) = U(\bar{g}, Q) = U(\bar{g}, P)$. But for all $\alpha \in [0, 1]$, $U(\alpha \bar{f} + (1 - \alpha)\bar{g}, P) = \alpha U(\bar{f}, P) + (1 - \alpha)U(\bar{g}, P)$ and therefore

$U(\alpha \bar{f} + (1 - \alpha)\bar{g}, P) = \alpha U(f, P) + (1 - \alpha)U(g, Q)$, proving that \mathcal{U} is convex. This also implies that $\mathcal{U} \times \mathcal{U}$ is convex.

Let $\mathcal{D} = \{(U(f, P), U(f, Q)) \mid f \in \mathcal{F}, P, Q \in \mathcal{P}\}$.

Lemma 1 $\mathcal{D} = \mathcal{U} \times \mathcal{U}$.

Proof Let $f, g \in \mathcal{F}$ and $P, Q \in \mathcal{P}$. By the same reasoning as above, there exist two constant acts \bar{f} and \bar{g} such that $U(f, P) = U(\bar{f}, P)$ and $U(g, Q) = U(\bar{g}, Q)$. Fix an arbitrary event $E \subset \Omega$ with $E \neq \Omega$, and let $P', Q' \in \mathcal{P}$ be such that $\text{supp}(P') \subseteq E$ and $\text{supp}(Q') \subseteq \Omega \setminus E$. We then have, by definition of U , $U(\bar{f}_E \bar{g}, P') = U(f, P)$ and $U(\bar{f}_E \bar{g}, Q') = U(g, Q)$, and therefore $(U(f, P), U(g, Q)) \in \mathcal{D}$, proving $\mathcal{D} = \mathcal{U} \times \mathcal{U}$. □

Now, define a binary relation \succeq on $\mathbb{U} = \mathcal{U} \times \mathcal{U}$ as follows: $(u, v) \succeq (u', v')$ if, and only if, there exist $f, g \in \mathcal{F}, P_1, P_2, Q_1, Q_2 \in \mathcal{P}$ such that $U(f, P_1) = u, U(f, P_2) = v, U(g, Q_1) = u', U(g, Q_2) = v'$ and $V(f, P_1, P_2) \geq V(g, Q_1, Q_2)$. By Lemma 1 and Axioms 1 and 2, \succeq is a well defined, complete, transitive and continuous binary relation on \mathbb{U} . Thus it can be represented by a continuous function $\hat{V} : \mathbb{U} \rightarrow \mathbb{R}$ (Debreu (1954), Theorem I). Moreover, by Axiom 2, $(u_1, u_2) \succeq (v_1, v_2)$ implies $(u_1, u_2) \succeq (v_1, v_2)$.¹¹ Thus \hat{V} is non decreasing. By definition, $V(f, P_1, P_2) \geq V(g, Q_1, Q_2)$ iff $\hat{V}(U(f, P_1), U(f, P_2)) \geq \hat{V}(U(g, Q_1), U(g, Q_2))$. Therefore, there exists an increasing function $W : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $f \in \mathcal{F}$ and $P_1, P_2 \in \mathcal{P}, V(f, P_1, P_2) = W(\hat{V}(U(f, P_1), U(f, P_2)))$. Let $\tilde{V} = W \circ \hat{V}$.

The two following steps essentially mimic Gilboa and Schmeidler (1989)'s and Chateauneuf (1991) proofs.

Lemma 2 For all $w \in \mathcal{D}, \alpha > 0$ such that $\alpha w \in \mathcal{D}, \tilde{V}(\alpha w) = \alpha \tilde{V}(w)$.

Proof Let \mathcal{F}^c the set of constant acts. Pick $f_0 \in \mathcal{F}^c$ such that $u(f_0(\omega)) = 0$ (this is possible, given the normalization we choose for V). Let $w = (w_1, w_2) \in \mathcal{D}$. By Lemma 1 there exist $f \in \mathcal{F}, Q_1, Q_2 \in \mathcal{P}$ such that $U(f, Q_1) = w_1$ and $U(f, Q_2) = w_2$. By definition of \tilde{V} and V we have $V(f, Q_1, Q_2) = \tilde{V}(w)$. For all $\alpha \in (0, 1)$, we have

$$\begin{aligned} U(\alpha f + (1 - \alpha)f_0, Q_1) &= \min_{p \in \psi(Q_1, Q_1)} \sum_{\omega} p(\omega) u(\alpha f(\omega) + (1 - \alpha)f_0(\omega)) \\ &= \min_{p \in \psi(Q_1, Q_1)} \sum_{\omega} p(\omega) (\alpha u(f(\omega)) + (1 - \alpha)u(f_0(\omega))) \\ &= \alpha U(f, P) \\ &= \alpha w_1. \end{aligned}$$

¹¹ For two vectors of real numbers $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we write $x \geq y$ whenever $x_1 \geq y_1$ and $x_2 \geq y_2$.

Similarly, $U(\alpha f + (1-\alpha)f_0, Q_2) = \alpha w_2$. Thus $\tilde{V}(\alpha w) = V(\alpha f + (1-\alpha)f_0, Q_1, Q_2)$. But:

$$\begin{aligned} V(\alpha f + (1-\alpha)f_0, Q_1, Q_2) &= \min_{p \in \psi(Q_1, Q_2)} \sum_{\omega} p(\omega) (u(\alpha f(\omega) + (1-\alpha)f_0(\omega))) \\ &= \min_{p \in \psi(Q_1, Q_2)} \sum_{\omega} p(\omega) (\alpha u(f(\omega) + (1-\alpha)f_0(\omega))) \\ &= \min_{p \in \psi(Q_1, Q_2)} \sum_{\omega} p(\omega) (\alpha u(f(\omega))), \\ &\quad \text{since } u(f_0(\omega)) = 0 \text{ for all } \omega \\ &= \alpha V(f, Q_1, Q_2), \text{ by definition of } V \\ &= \alpha \tilde{V}(w). \end{aligned}$$

Thus $\tilde{V}(\alpha w) = \alpha \tilde{V}(w)$ for all $\alpha \in [0, 1]$, and thus for all $\alpha > 0$. □

We extend \tilde{V} to \mathbb{R}^2 by homogeneity, and still call \tilde{V} its extension (which is homogeneous and monotone).

Lemma 3 For all $w \in \mathbb{R}^2, \mu \in \mathbb{R}, \tilde{V}(w + (\mu, \mu)) = \tilde{V}(w) + \mu$.

Proof Let w_1, w_2, μ be such that $2w_1, 2w_2, 2\mu \in \mathcal{W}$. Given the homogeneity of \tilde{V} , this assumption is without any loss of generality. Let $f \in \mathcal{F}, h \in \mathcal{F}^c$ and $P_1, P_2 \in \mathcal{P}$ be such that $U(f, P_1) = 2w_1, U(f, P_2) = 2w_2$, and $U(h, P_1) = 2\mu$. Note that since h is constant, we actually have $U(h, P_1) = U(h, P_2) = V(h, P_1, P_2) = 2\mu$. We obtain:

$$\begin{aligned} \tilde{V}(w + (\mu, \mu)) &= \tilde{V}\left(\frac{1}{2}U(f, P_1) + \frac{1}{2}U(h, P_1), \frac{1}{2}U(f, P_2) + \frac{1}{2}U(h, P_2)\right) \\ &= \tilde{V}\left(U\left(\frac{1}{2}f + \frac{1}{2}h, P_1\right), U\left(\frac{1}{2}f + \frac{1}{2}h, P_2\right)\right) \\ &= V\left(\frac{1}{2}f + \frac{1}{2}h, P_1, P_2\right) \\ &= \frac{1}{2}V(f, P_1, P_2) + \frac{1}{2}V(h, P_1, P_2) \\ &= \frac{1}{2}\tilde{V}(2w) + \mu \\ &= \tilde{V}(w) + \mu. \end{aligned}$$

□

It remains to show that \tilde{V} is symmetric and concave. This is the key part of the proof, where the Disagreement Aversion axiom plays a crucial role.

Lemma 4 \tilde{V} is symmetric.

Proof We first show that \tilde{V} is symmetric. Without loss of generality (because of the homogeneity of \tilde{V}), choose $w, w' \in \mathcal{D}$. Let $f \in \mathcal{F}$ and $P_1, P_2 \in \mathcal{P}$ be such that $U(f, P_1) = w_1$ and $U(f, P_2) = w_2$. Let $\alpha \in [0, 1]$. We have:

$$\tilde{V}(\alpha w_1 + (1-\alpha)w_2, \alpha w_2 + (1-\alpha)w_1)$$

$$\begin{aligned}
 &= \tilde{V}(\alpha U(f, P_1) + (1 - \alpha)U(f, P_2), \alpha U(f, P_2) + (1 - \alpha)U(f, P_1)) \\
 &= \tilde{V}(U(f, \alpha P_1 + (1 - \alpha)P_2), U(\alpha P_2 + (1 - \alpha)P_1)), \\
 &\text{by linearity of } \psi \text{ on } \{(P, P) : P \in \mathcal{P}\}.
 \end{aligned}$$

Thus Axiom 3 implies for all $w_1, w_2 \in \mathbb{R}$ (by homogeneity) and all $\alpha \in [0, 1]$:

$$\tilde{V}(\alpha w_1 + (1 - \alpha)w_2, \alpha w_2 + (1 - \alpha)w_1) \geq \tilde{V}(w_1, w_2).$$

Thus, in particular, \tilde{V} must be symmetric. Indeed, setting $\alpha = 0$, we obtain $\tilde{V}(w_2, w_1) \geq \tilde{V}(w_1, w_2)$. Permuting w_1 and w_2 yields $\tilde{V}(w_1, w_2) \geq \tilde{V}(w_2, w_1)$, and thus $\tilde{V}(w_1, w_2) = \tilde{V}(w_2, w_1)$. □

Lemma 5 \tilde{V} is concave.

Proof Let $w = (w_1, w_2)$ and $w' = (w'_1, w'_2)$ in \mathcal{D} . Let us prove that for all $\alpha \in [0, 1]$,

$$\begin{aligned}
 \tilde{V}(\alpha w + (1 - \alpha)w') &= \tilde{V}(\alpha w + (1 - \alpha)(\theta w + (1 - \theta)\bar{w})) \\
 &\geq \alpha \tilde{V}(w) + (1 - \alpha)\tilde{V}(w').
 \end{aligned}$$

We will first suppose that $\tilde{V}(w) = \tilde{V}(w')$, and let consider two subcases: $w_1 \leq w_2, w'_1 \leq w'_2$ (Step 1) and $w_1 \leq w_2, w'_1 \geq w'_2$ (Step 2). We will finally consider the case $\tilde{V}(w) \neq \tilde{V}(w')$ in Step 3.

Step 1.

Let $w = (w_1, w_2)$ and $w' = (w'_1, w'_2)$ in \mathcal{D} be such that $\tilde{V}(w) = \tilde{V}(w')$, $w_1 \leq w_2, w'_1 \leq w'_2$, and $\bar{w} = (t, t) \in \mathcal{D}$ be such that $\tilde{V}(\bar{w}) = \tilde{V}(w)$ (by axioms continuity and monotonicity of \tilde{V} , such an \bar{w} exists). Without loss of generality, assume $0 < w_1 \leq w'_1$ (and thus, by Axiom 2, $w_2 \geq w'_2$) and $0 < w'_2$. First, we show that there exists $\theta \in [0, 1]$ such that $w' = \theta w + (1 - \theta)\bar{w}$. Assume that such is not the case. Let $\lambda > 0$ be such that $\lambda w' \in [w, \bar{w}]$, and let $\hat{\theta}$ be such that $\lambda w' = \hat{\theta} w + (1 - \hat{\theta})\bar{w}$. Since $w' \notin [w, \bar{w}]$, $\lambda \neq 1$. Thus, by Axiom 2, either $\tilde{V}(\lambda w') > \tilde{V}(w')$ or $\tilde{V}(\lambda w') < \tilde{V}(w')$. But by Lemmas 2 and 3, $\tilde{V}(\lambda w') = \tilde{V}(\hat{\theta} w + (1 - \hat{\theta})\bar{w}) = \hat{\theta} \tilde{V}(w) + (1 - \hat{\theta})\tilde{V}(\bar{w}) = \tilde{V}(w) = \tilde{V}(w')$, a contradiction.

Thus, let θ be such that $w' = \theta w + (1 - \theta)\bar{w}$. Then, for all $\alpha \in [0, 1]$,

$$\begin{aligned}
 \tilde{V}(\alpha w + (1 - \alpha)w') &= \tilde{V}(\alpha w + (1 - \alpha)(\theta w + (1 - \theta)\bar{w})) \\
 &= \tilde{V}((\alpha + (1 - \alpha)\theta)w + (1 - \alpha)(1 - \theta)\bar{w}) \\
 &= (\alpha + (1 - \alpha)\theta)\tilde{V}(w) + (1 - \alpha)(1 - \theta)\tilde{V}(\bar{w}) \\
 &= \tilde{V}(w) \\
 &= \alpha \tilde{V}(w) + (1 - \alpha)\tilde{V}(w'),
 \end{aligned}$$

where the third equality follows by c -affinity and homogeneity of \tilde{V} .

Step 2.

Assume now that $w = (w_1, w_2)$ and $w' = (w'_1, w'_2)$ in \mathcal{D} are such that $\tilde{V}(w) = \tilde{V}(w')$, $w_1 \leq w_2, w'_1 \geq w'_2$, and let $\bar{w} = (t, t) \in \mathcal{D}$ be such that $\tilde{V}(\bar{w}) = \tilde{V}(w)$. We

assume, without loss of generality, $0 < w_1 \leq w'_2$ (and thus, by Axiom 2 and Lemma 4, $w_2 \geq w'_2$) and $0 < w'_2$.

By Lemma 4, $\tilde{V}(w_1, w_2) = \tilde{V}(w_2, w_1)$. Thus $\tilde{V}(w_2, w_1) = \tilde{V}(w'_1, w'_2)$. By the preceding argument, there exists $\theta \in [0, 1]$ such that $w' = \theta(w_2, w_1) + (1 - \theta)\bar{w}$. Therefore, for all $\alpha \in [0, 1]$:

$$\begin{aligned} \tilde{V}(\alpha w + (1 - \alpha)w') &= \tilde{V}(\alpha w + (1 - \alpha)(\theta(w_2, w_1) + (1 - \theta)\bar{w})) \\ &= \tilde{V}((\alpha w + (1 - \alpha)\theta(w_2, w_2)) + (1 - \alpha)(1 - \theta)\bar{w}) \\ &= (\alpha + (1 - \alpha)\theta)\tilde{V}\left(\frac{\alpha}{\alpha + (1 - \alpha)\theta}w + \frac{1 - \alpha}{\alpha + (1 - \alpha)\theta}(w_2, w_1)\right) \\ &\quad + (1 - \alpha)(1 - \theta)\tilde{V}(\bar{w}), \text{ by } c\text{-affinity and homogeneity of } \tilde{V} \\ &\geq (\alpha + (1 - \alpha)\theta)\tilde{V}(w) + (1 - \alpha)(1 - \theta)\tilde{V}(\bar{w}), \text{ by Axiom 3,} \end{aligned}$$

and therefore $\tilde{V}(\alpha w + (1 - \alpha)w') \geq \tilde{V}(w)$, the desired result.

Step 3.

It remains to deal with the case where $\tilde{V}(w) \neq \tilde{V}(w')$. Assume without loss of generality that $\tilde{V}(w) > \tilde{V}(w')$. Let $\mu = \tilde{V}(w) - \tilde{V}(w')$. Define $\tilde{w} = w' + (\mu, \mu)$. By c -affinity of \tilde{V} , we have $\tilde{V}(\tilde{w}) = \tilde{V}(w') + \mu = \tilde{V}(w)$. Thus, for all $\alpha \in [0, 1]$,

$$\begin{aligned} \tilde{V}(\alpha\tilde{w} + (1 - \alpha)w) &\geq \alpha\tilde{V}(\tilde{w}) + (1 - \alpha)\tilde{V}(w), \text{ by steps 1 and 2} \\ &\geq \alpha\tilde{V}(w') + (1 - \alpha)\tilde{V}(w) + \alpha\mu. \end{aligned}$$

On the other hand,

$$\begin{aligned} \tilde{V}(\alpha\tilde{w} + (1 - \alpha)w) &= \tilde{V}(\alpha(w' + \mu) + (1 - \alpha)w) \\ &= \tilde{V}(\alpha w' + (1 - \alpha)w) + \alpha\mu \text{ by } c\text{-affinity of } \tilde{V}. \end{aligned}$$

Therefore $\tilde{V}(\alpha w' + (1 - \alpha)w) \geq \alpha\tilde{V}(w') + (1 - \alpha)\tilde{V}(w)$, the desired result. \square

By Lemmas 2, 3 and 5, \tilde{V} is concave and homogeneous of degree 1, and c -affine. Therefore, by a classical result (see, e.g., the ‘‘Fundamental Lemma’’ in Chateauneuf (1991) and Lemma 3.5 in Gilboa and Schmeidler (1989)), there exists a unique closed and convex set Π such that $\tilde{V}(w_1, w_2) = \min_{\pi \in \Pi} \pi(1)w_1 + \pi(2)w_2$. Furthermore, by Lemma 4, Π is symmetric. Recall that by definition $V(f, P_1, P_1) \geq V(g, Q_1, Q_1)$ iff $\tilde{V}(U(f, P_1), U(f, P_2)) \geq \tilde{V}(U(g, Q_1), U(f, Q_2))$. Then, the definition of U and Axiom 1 yields to Theorem 1, with $\varphi(P) = \psi(P, P)$.

A.3 Proof of Proposition 2

This proposition is in the vein of Theorem 3 in Gajdos et al. (2004) and Theorem 4 in Gajdos et al. (2008a). For sake of exactness, we adapt the proof here.

It is straightforward to check that 2 implies 1.

Conversely, suppose ad absurdum that \succ_b is more averse to imprecision than \succ_a but that there exists $p^* \in \varphi_a(P)$ such that $p \notin \varphi_b(P)$. Using a separation argument,

there exists a function $\phi : \Omega \rightarrow \mathbb{R}$ such that $E_{p^*}\phi < \min_{p \in \varphi_b(P)} E_p\phi$. Let \bar{x} and \underline{x} in X be such that both a and b strictly prefer \bar{x} to \underline{x} . Note that we can choose by normalization u_a and u_b so that $u_a(\bar{x}) = u_b(\bar{x}) = 1 > u_a(\underline{x}) = u_b(\underline{x}) = 0$. Since Ω is a finite set, there exist numbers $m > 0$ and ℓ , such that for all $\omega \in \Omega$, $m\phi(\omega) + \ell \in [0, 1]$. Let $\alpha_\omega = m\phi(\omega) + \ell$, $\omega \in \Omega$. Let $f^0 \in \mathcal{F}_{\bar{x}, \underline{x}}^b$ such that $f^0(\omega) = \alpha_\omega\delta_{\bar{x}} + (1 - \alpha_\omega)\delta_{\underline{x}}$ for all $\omega \in \Omega$. Then, $E_{p^*}u(f^0) < \min_{p \in \varphi_b(P)} E_pu(f^0)$ which implies that $(f, P, P) \succ_b (f, \{p^*\}, \{p^*\})$. However, since $p^* \in \varphi_a(P)$, $E_{p^*}u(f^0) \geq \min_{p \in \varphi_a(P)} E_pu(f^0)$ which implies that $(f, \{p^*\}, \{p^*\}) \succ_a (f, P, P)$ and thus yields a contradiction with \succ_b being more averse to imprecision than \succ_a .

A.4 Proof of Proposition 3

Since Π_a and Π_b are symmetric, there exist α_a and α_b such that $\Pi_a = \{(1 - \alpha_a)(\frac{1}{2}, \frac{1}{2}) + \alpha_a(t, 1 - t) | t \in [0, 1]\}$ and $\Pi_b = \{(1 - \alpha_b)(\frac{1}{2}, \frac{1}{2}) + \alpha_b(t, 1 - t) | t \in [0, 1]\}$. For all $f \in \mathcal{F}$, $P, Q \in \mathcal{P}$, $\alpha \in (0, 1)$, $i = a, b$, $(f, \alpha P + (1 - \alpha)Q, \alpha P + (1 - \alpha)Q) \succ_i (f, P, Q)$ if:

$$\begin{aligned} & \min_{p \in \varphi_i(\alpha P + (1 - \alpha)Q)} E_pu(f) \\ & \geq \min_{(\pi, 1 - \pi) \in \Pi_i} \left(\pi \min_{p \in \varphi_i(P)} E_pu(f) + (1 - \pi) \min_{p \in \varphi_i(Q)} E_pu(f) \right). \end{aligned}$$

Since the φ_i are linear, we then obtain:

$$\begin{aligned} & \alpha \min_{p \in \varphi_i(P)} E_pu(f) + (1 - \alpha) \min_{p \in \varphi_i(Q)} E_pu(f) \\ & \geq \left(\frac{1 + \alpha_i}{2} \right) \min \left(\min_{p \in \varphi_i(P)} E_pu(f), \min_{p \in \varphi_i(Q)} E_pu(f) \right) \\ & \quad + \left(\frac{1 - \alpha_i}{2} \right) \max \left(\min_{p \in \varphi_i(P)} E_pu(f), \min_{p \in \varphi_i(Q)} E_pu(f) \right) \end{aligned}$$

Furthermore, if both a and b prefer (f, P) to (f, Q) , then $(f, \alpha P + (1 - \alpha)Q, \alpha P + (1 - \alpha)Q) \succ_i (f, P, Q)$ iff $\alpha \geq \frac{1 - \alpha_i}{2}$.

Thus if \succ_b is more averse to disagreement than \succ_a , then $\frac{1 - \alpha_a}{2} \geq \frac{1 - \alpha_b}{2}$ and thus $\Pi_a \subseteq \Pi_b$. Conversely, if $\Pi_a \subseteq \Pi_b$, then \succ_b is more averse to disagreement than \succ_a .

A.5 Proof of Proposition 4

Since for all $P \in \mathcal{P}$, $\varphi(P) \subseteq P$, we have, for all $s \in \Omega$, $\varphi(\{\delta_s\}) = \delta_s$. The Conflict Aversion Hypothesis implies $\Pi \otimes (\varphi(\Delta_{st}), \varphi(\Delta_{st})) \subseteq \Pi \otimes (\varphi(\{\delta_s\}), \varphi(\{\delta_t\}))$, and thus $\varphi(\Delta_{st}) \subseteq \Pi \otimes (\{\delta_s\}, \{\delta_t\})$. Axiom (Reduction) implies that there exists $R \subseteq \Delta_{st}$ such that $\Pi \otimes (\{\delta_s\}, \{\delta_t\}) = \varphi(R)$. Thus we have:

$$\varphi(\Delta_{st}) \subseteq \Pi \otimes (\{\delta_s\}, \{\delta_t\}) = \varphi(R),$$

and therefore $\varphi(\Delta_{st}) \subseteq \varphi(R)$. Since $R \subseteq \Delta_{st}$, Axiom (Preference for Precision of Information) implies $\varphi(R) = \varphi(\Delta_{st})$. Therefore, $\Pi \otimes (\varphi(\{\delta_s\}), \varphi(\{\delta_t\})) = \varphi(\Delta_{st})$, and thus $(f, \{\delta_s\}, \{\delta_t\}) \sim (f, \Delta_{st}, \Delta_{st})$ for all $f \in \mathcal{F}$, which contradicts the Conflict Aversion Hypothesis.

A.6 Proof of Example 3

Assume without loss of generality that $\Omega = \{1, 2\}$. Let $\delta_1 = (1, 0)$, $\delta_2 = (0, 1)$ and $\delta_{12} = \{p = (p_1, p_2) \in [0, 1]^2 \mid p_1 + p_2 = 1\}$. Any set $P \in \mathcal{P}$ can be written in a unique way as: $P = \gamma_1\delta_1 + \gamma_2\delta_2 + \gamma_3\delta_{12}$, with $\gamma_1, \gamma_2, \gamma_3 \geq 0$ such that $\gamma_1 + \gamma_2 + \gamma_3 = 1$.

For all $R = \gamma_1\delta_1 + \gamma_2\delta_2 + \gamma_3\delta_{12}$, and all $\theta \in [0, 1]$, we have:

$$\varphi(R) = \left(\gamma_1 + \frac{(1-\theta)\gamma_3}{2}\right)\delta_1 + \left(\gamma_2 + \frac{(1-\theta)\gamma_3}{2}\right)\delta_2 + \theta\gamma_3\delta_{12}.$$

(1 \Rightarrow 2).

Let us suppose that Axiom (Reduction) holds, that is for all $P, Q \subseteq \Delta(\{1, 2\})$, there exists $R \subseteq \Delta(\{1, 2\})$ such that $\Pi \otimes (\varphi(P), \varphi(Q)) = \varphi(R)$.

Let $P = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_{12}$, $Q = \lambda'_1\delta_1 + \lambda'_2\delta_2 + \lambda'_3\delta_{12}$ and $\alpha \in [0, 1], \theta \in [0, 1]$. Without any loss of generality, let us assume that $\lambda'_1 + \frac{1-\theta}{2}\lambda'_3 \geq \lambda_1 + \frac{1-\theta}{2}\lambda_3$.

Consider a first case where $\theta = 0$. Then

$$\begin{aligned} \varphi(P) &= \left(\lambda_1 + \frac{\lambda_3}{2}\right)\delta_1 + \left(\lambda_2 + \frac{\lambda_3}{2}\right)\delta_2, \\ \varphi(Q) &= \left(\lambda'_1 + \frac{\lambda'_3}{2}\right)\delta_1 + \left(\lambda'_2 + \frac{\lambda'_3}{2}\right)\delta_2. \end{aligned}$$

Simple computations show that

$$\begin{aligned} &\Pi \otimes (a\delta_1 + (1-a)\delta_2, b\delta_1 + (1-b)\delta_2) \\ &= \left(\frac{1+\alpha}{2}a + \frac{1-\alpha}{2}b\right)\delta_1 + \left(\frac{1-\alpha}{2}a + \frac{1+\alpha}{2}b\right)\delta_2 + (b-a)\alpha\delta_{12} \end{aligned}$$

if $a \leq b$.

If R is such that $\varphi(R) = \Pi \otimes (\varphi(P), \varphi(Q))$, then we must have

$$\begin{aligned} \varphi(R) &= \left(\gamma_1 + \frac{\gamma_3}{2}\right)\delta_1 + \left(\gamma_2 + \frac{\gamma_3}{2}\right)\delta_2 = \Pi \otimes (\varphi(P), \varphi(Q)) \\ &= \left(\frac{1+\alpha}{2}a + \frac{1-\alpha}{2}b\right)\delta_1 + \left(\frac{1-\alpha}{2}a + \frac{1+\alpha}{2}b\right)\delta_2 + (b-a)\alpha\delta_{12}, \end{aligned}$$

where $a = \lambda_1 + \frac{\lambda_3}{2}$ and $b = \lambda'_1 + \frac{\lambda'_3}{2}$.

Therefore, we must have $\alpha = 0$ and thus $\theta \geq \alpha$.

Let suppose now that $\theta > 0$. Consider the case where $\lambda_2 = 1$ and $\lambda'_1 = 1$. Then:

$$\begin{aligned} \Pi \otimes (\varphi(P), \varphi(Q)) &= \Pi \otimes (\delta_2, \delta_1) \\ &= \frac{1-\alpha}{2}\delta_1 + \frac{1-\alpha}{2}\delta_2 + \alpha\delta_{12}. \end{aligned}$$

If R is such that $\varphi(R) = \Pi \otimes (\varphi(P), \varphi(Q))$, then we must have:

$$\begin{aligned} \varphi(R) &= \left(\gamma_1 + \frac{(1-\theta)\gamma_3}{2}\right)\delta_1 + \left(\gamma_2 + \frac{(1-\theta)\gamma_3}{2}\right)\delta_2 + \theta\gamma_3\delta_{12} \\ &= \Pi \otimes (\varphi(P), \varphi(Q)) = \frac{1-\alpha}{2}\delta_1 + \frac{1-\alpha}{2}\delta_2 + \alpha\delta_{12}. \end{aligned}$$

Thus $\gamma_3 = \frac{\alpha}{\theta}$. Since $\gamma_3 \leq 1$, $\theta \geq \alpha$.

(2 \Rightarrow 1)

Let us consider a first case where $\theta = \alpha = 0$. Then $R = \Pi \otimes (P, Q)$ is such that $\varphi(R) = \Pi \otimes (\varphi(P), \varphi(Q))$ and thus for all $P, Q \in \mathcal{P}$, there exists $R \in \mathcal{P}$ such that $\Pi \otimes (\varphi(P), \varphi(Q)) = \varphi(R)$.

Assume that $\theta \geq \alpha$ and $\theta > 0$ and consider two subcases.

Case 1: $\lambda'_2 + \frac{(1-\theta)\lambda'_3}{2} \geq \lambda_2 + \frac{(1-\theta)\lambda_3}{2}$ (it corresponds to the case where $\varphi(Q) \subseteq \varphi(P)$).

Simple computations show that

$$\begin{aligned} &\Pi \otimes (a_1\delta_1 + a_2\delta_2 + (1-a_1-a_2)\delta_{12}, b_1\delta_1 + b_2\delta_2 + (1-b_1-b_2)\delta_{12}) \\ &= \left(\frac{1+\alpha}{2}a_1 + \frac{1-\alpha}{2}b_1\right)\delta_1 + \left(\frac{1+\alpha}{2}a_2 + \frac{1-\alpha}{2}b_2\right)\delta_2 \\ &\quad + \left[\left(\frac{1+\alpha}{2}\right)(1-a_1-a_2) + \left(\frac{1-\alpha}{2}\right)(1-b_1-b_2)\right]\delta_{12} \end{aligned}$$

if $a_1 \leq b_1$ and $a_2 \leq b_2$.

If R is such that $\varphi(R) = \Pi \otimes (\varphi(P), \varphi(Q))$, we must have

$$\begin{aligned} \varphi(R) &= \left(\gamma_1 + \frac{(1-\theta)\gamma_3}{2}\right)\delta_1 + \left(\gamma_2 + \frac{(1-\theta)\gamma_3}{2}\right)\delta_2 + \theta\gamma_3\delta_{12} \\ &= \Pi \otimes (\varphi(P), \varphi(Q)) \\ &= \left(\frac{1+\alpha}{2}a_1 + \frac{1-\alpha}{2}b_1\right)\delta_1 + \left(\frac{1+\alpha}{2}a_2 + \frac{1-\alpha}{2}b_2\right)\delta_2 \\ &\quad + \left[\left(\frac{1+\alpha}{2}\right)(1-a_1-a_2) + \left(\frac{1-\alpha}{2}\right)(1-b_1-b_2)\right]\delta_{12} \end{aligned}$$

where $a_1 = \lambda_1 + \frac{1-\theta}{2}\lambda_3$, $a_2 = \lambda_2 + \frac{(1-\theta)}{2}\lambda_3$, $b_1 = \lambda'_1 + \frac{1-\theta}{2}\lambda'_3$ and $b_2 = \lambda'_2 + \frac{(1-\theta)}{2}\lambda'_3$.

Therefore, we have:

$$\begin{aligned}\gamma_1 + \frac{(1-\theta)\gamma_3}{2} &= \frac{1+\alpha}{2}a_1 + \frac{1-\alpha}{2}b_1 \\ \gamma_2 + \frac{(1-\theta)\gamma_3}{2} &= \frac{1+\alpha}{2}a_2 + \frac{1-\alpha}{2}b_2 \\ \theta\gamma_3 &= \left(\frac{1+\alpha}{2}\right)(1-a_1-a_2) + \left(\frac{1-\alpha}{2}\right)(1-b_1-b_2),\end{aligned}$$

which leads to

$$\begin{aligned}\gamma_1 &= \frac{1+\alpha}{2}\lambda_1 + \frac{1-\alpha}{2}\lambda'_1 \\ \gamma_2 &= \frac{1+\alpha}{2}\lambda_2 + \frac{1-\alpha}{2}\lambda'_2 \\ \gamma_3 &= \left(\frac{1+\alpha}{2}\right)\lambda_3 + \left(\frac{1-\alpha}{2}\right)\lambda'_3,\end{aligned}$$

which are three values between 0 and 1. In fact, when $Q \subseteq P$, there always exists R such that $\varphi(R) = \Pi \otimes (\varphi(P), \varphi(Q))$ as soon as $\theta > 0$. R is simply:

$$R = \left(\frac{1+\alpha}{2}\right)P + \left(\frac{1-\alpha}{2}\right)Q.$$

Case 2: $\lambda_2 + \frac{(1-\theta)\lambda_3}{2} \geq \lambda'_2 + \frac{(1-\theta)\lambda'_3}{2}$.

Simple computation gives that

$$\begin{aligned}\Pi \otimes (a_1\delta_1 + a_2\delta_2 + (1-a_1-a_2)\delta_{12}, b_1\delta_1 + b_2\delta_2 + (1-b_1-b_2)\delta_{12}) \\ = \left(\frac{1+\alpha}{2}a_1 + \frac{1-\alpha}{2}b_1\right)\delta_1 + \left(\frac{1-\alpha}{2}a_2 + \frac{1+\alpha}{2}b_2\right)\delta_2 \\ + \left[\left(\frac{1+\alpha}{2}\right)(1-a_1-b_2) + \left(\frac{1-\alpha}{2}\right)(1-b_1-a_2)\right]\delta_{12}\end{aligned}$$

if $a_1 \leq b_1$ and $a_2 \geq b_2$.

If R is such that $\varphi(R) = \Pi \otimes (\varphi(P), \varphi(Q))$, then we must have

$$\begin{aligned}\varphi(R) &= \left(\gamma_1 + \frac{(1-\theta)\gamma_3}{2}\right)\delta_1 + \left(\gamma_2 + \frac{(1-\theta)\gamma_3}{2}\right)\delta_2 + \theta\gamma_3\delta_{12} \\ &= \Pi \otimes (\varphi(P), \varphi(Q)) \\ &= \left(\frac{1+\alpha}{2}a_1 + \frac{1-\alpha}{2}b_1\right)\delta_1 + \left(\frac{1-\alpha}{2}a_2 + \frac{1+\alpha}{2}b_2\right)\delta_2 \\ &\quad + \left[\left(\frac{1+\alpha}{2}\right)(1-a_1-b_2) + \left(\frac{1-\alpha}{2}\right)(1-b_1-a_2)\right]\delta_{12}\end{aligned}$$

where $a_1 = \lambda_1 + \frac{1-\theta}{2}\lambda_3$, $a_2 = \lambda_2 + \frac{(1-\theta)}{2}\lambda_3$, $b_1 = \lambda'_1 + \frac{1-\theta}{2}\lambda'_3$ and $b_2 = \lambda'_2 + \frac{(1-\theta)}{2}\lambda'_3$.

To show that there exists R , we first show that we can find γ_3 such that $1 \geq \gamma_3 \geq 0$ and such that

$$\theta\gamma_3 = \left(\frac{1+\alpha}{2}\right)(1-a_1-b_2) + \left(\frac{1-\alpha}{2}\right)(1-b_1-a_2).$$

Then we must have:

$$\gamma_3 = \frac{1}{2} \left[\lambda_3 + \lambda'_3 + \frac{\alpha}{\theta}(\lambda'_1 - \lambda_1 + \lambda_2 - \lambda'_2) \right].$$

Since $\lambda'_1 + \frac{1-\theta}{2}\lambda'_3 \geq \lambda_1 + \frac{1-\theta}{2}\lambda_3$ and $\lambda_2 + \frac{(1-\theta)\lambda_3}{2} \geq \lambda'_2 + \frac{(1-\theta)\lambda'_3}{2}$ we have:

$$\begin{aligned} \lambda'_1 - \lambda_1 &\geq \frac{1-\theta}{2}(\lambda_3 - \lambda'_3) \\ \lambda_2 - \lambda'_2 &\geq \frac{1-\theta}{2}(\lambda'_3 - \lambda_3) \end{aligned}$$

and thus

$$\lambda'_1 - \lambda_1 + \lambda_2 - \lambda'_2 \geq 0.$$

Therefore $\gamma_3 \geq 0$.

On the other hand, since $\theta \geq \alpha$,

$$\gamma_3 \leq \frac{1}{2} [\lambda_3 + \lambda'_3 + \lambda'_1 - \lambda_1 + \lambda_2 - \lambda'_2] = [\lambda_2 + \lambda_3 + \lambda'_1 + \lambda'_3 - 1] \leq 1.$$

That means that we can always find a γ_3 that fits for the size of the probability interval $\Pi \otimes (\varphi(P), \varphi(Q))$.

It remains to be shown that we can find γ_1 and γ_2 that can adjust for the margins. Let χ be the size of the probability interval $\Pi \otimes (\varphi(P), \varphi(Q))$. Once χ is fixed, and thus $\gamma_3 = \frac{\chi}{\theta}$, we can find γ_1 and γ_2 values such that $\varphi(R) = \tau\delta_1 + (1 - \chi - \tau)\delta_2 + \chi\delta_{12}$ for any $\tau \in \left[\frac{(1-\theta)\chi}{2}, 1 - \frac{\chi}{\theta} + \frac{(1-\theta)\chi}{2} \right]$.

On the other hand, the weight on δ_1 in the decomposition of $\Pi \otimes (\varphi(P), \varphi(Q))$ is equal to $\frac{1+\alpha}{2}(\lambda_1 + \frac{1-\theta}{2}\lambda_3) + \frac{1-\alpha}{2}(\lambda'_1 + \frac{1-\theta}{2}\lambda'_3)$ with

$$\chi = \frac{1}{2} [\theta(\lambda_3 + \lambda'_3) + \alpha(\lambda'_1 - \lambda_1 + \lambda_2 - \lambda'_2)].$$

For χ fixed, to minimize $\tau' = \frac{1+\alpha}{2}(\lambda_1 + \frac{1-\theta}{2}\lambda_3) + \frac{1-\alpha}{2}(\lambda'_1 + \frac{1-\theta}{2}\lambda'_3)$, we have to consider $\lambda_1 = \lambda'_1 = 0$. Then

$$\chi = \frac{1}{2} [\theta(\lambda_3 + \lambda'_3) + \alpha(\lambda'_3 - \lambda_3)],$$

while

$$\begin{aligned} \tau' &= \frac{1 + \alpha}{2} \frac{1 - \theta}{2} \lambda_3 + \frac{1 - \alpha}{2} \frac{1 - \theta}{2} \lambda'_3 \\ &= \frac{1 - \theta}{2} \frac{1}{2} (\lambda_3 + \lambda'_3 + \alpha(\lambda_3 - \lambda'_3)) \\ &= \frac{1 - \theta}{2} \frac{1}{2} \frac{1}{\theta} (\theta (\lambda_3 + \lambda'_3) + \alpha(\lambda'_3 - \lambda_3) + \alpha (\theta(\lambda_3 - \lambda'_3) - (\lambda'_3 - \lambda_3))) \\ &= \frac{(1 - \theta)}{2} \frac{\chi}{\theta} + \frac{1 - \theta}{2} \frac{1}{2} \frac{\alpha}{\theta} (1 - \theta)(\lambda'_3 - \lambda_3). \end{aligned}$$

Since we have also $\lambda'_1 + \frac{1-\theta}{2} \lambda'_3 \geq \lambda_1 + \frac{1-\theta}{2} \lambda_3$, then $\tau' \geq \frac{(1-\theta)}{2} \frac{\chi}{\theta}$: for a fixed χ , the lowest weight on δ_1 that can be observed for $\Pi \otimes (\varphi(P), \varphi(Q))$ can be obtained by some γ_1 and γ_2 . Using a similar proof, the same result holds for the lowest weight on δ_2 that can be observed for $\Pi \otimes (\varphi(P), \varphi(Q))$, which means that the highest weight on δ_1 that can be observed for $\Pi \otimes (\varphi(P), \varphi(Q))$ can also be obtained by some γ_1 and γ_2 .

Therefore, there exists R such that $\varphi(R) = \Pi \otimes (\varphi(P), \varphi(Q))$.

(2 \Leftrightarrow 3)

Let us consider $\{p_1\}, \{p_2\} \subseteq \Delta(\{1, 2\})$. To prove the equivalence, it is sufficient to prove the equivalence between

$$\theta \geq \alpha \Leftrightarrow \Pi \otimes (\varphi(\{p_1\}), \varphi(\{p_2\})) \subseteq \Pi \otimes (\varphi(\text{co}\{p_1, p_2\}), \varphi(\text{co}\{p_1, p_2\}))$$

Let $p_1 = \lambda_1 \delta_1 + \lambda_2 \delta_2, p_2 = \lambda'_1 \delta_1 + \lambda'_2 \delta_2$ and suppose without loss of generality that $\lambda_1 \leq \lambda'_1$.

We have that $\text{co}\{p_1, p_2\} = \lambda_1 \delta_1 + \lambda'_2 \delta_2 + (\lambda'_1 - \lambda_1) \delta_{12}$ and thus

$$\begin{aligned} \Pi \otimes (\varphi(\text{co}\{p_1, p_2\}), \varphi(\text{co}\{p_1, p_2\})) &= \varphi(\text{co}\{p_1, p_2\}) \\ &= \left(\lambda_1 + \frac{(1 - \theta)(\lambda'_1 - \lambda_1)}{2} \right) \delta_1 + \left(\lambda_2 + \frac{(1 - \theta)(\lambda'_1 - \lambda_1)}{2} \right) \delta_2 \\ &\quad + \theta (\lambda'_1 - \lambda_1) \delta_{12}. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} \Pi \otimes (\varphi(\{p_1\}), \varphi(\{p_2\})) &= \Pi \otimes (\lambda_1 \delta_1 + \lambda_2 \delta_2, \lambda'_1 \delta_1 + \lambda'_2 \delta_2) \\ &= \left(\lambda_1 + \frac{(1 - \alpha)(\lambda'_1 - \lambda_1)}{2} \right) \delta_1 \\ &\quad + \left(\lambda_2 + \frac{(1 - \alpha)(\lambda'_1 - \lambda_1)}{2} \right) \delta_2 + \alpha (\lambda'_1 - \lambda_1) \delta_{12} \end{aligned}$$

Then

$$\begin{aligned} \Pi \otimes (\varphi(P), \varphi(Q)) &= \Pi \otimes (\delta_1, \delta_2) \\ &= \left(\frac{1-\alpha}{2}\right) \delta_1 + \left(\frac{1-\alpha}{2}\right) \delta_2 + \alpha \delta_{12}. \end{aligned}$$

Thus $\Pi \otimes (\varphi(\{p_1\}), \varphi(\{p_2\})) \subseteq \Pi \otimes (\varphi(\text{co}\{p_1, p_2\}), \varphi(\text{co}\{p_1, p_2\}))$ iff

$$\begin{aligned} \lambda_1 + \frac{(1-\alpha)(\lambda'_1 - \lambda_1)}{2} &\geq \lambda_1 + \frac{(1-\theta)(\lambda'_1 - \lambda_1)}{2} \\ \lambda_2 + \frac{(1-\alpha)(\lambda'_1 - \lambda_1)}{2} &\geq \lambda_2 + \frac{(1-\theta)(\lambda'_1 - \lambda_1)}{2} \\ \alpha(\lambda'_1 - \lambda_1) &\leq \theta(\lambda'_1 - \lambda_1) \end{aligned}$$

and thus iff $\theta \geq \alpha$.

A.7 Proof of Proposition 5

It is obvious that (ii) implies (i). We show the converse implication. Assume that (i) holds. This implies, for all $P, Q \in \mathcal{P}$, $\Pi \otimes (\varphi(P), \varphi(Q)) = \varphi(\text{co}(P \cup Q))$. Observe that, since $\varphi(P) \subseteq P$ for all P , we have $\varphi(\{p\}) = \{p\}$ for all $p \in \Delta(\Omega)$. Thus we have $\Pi \otimes (\{q\}, \varphi(Q)) = \varphi(Q)$ for all $q \in Q$. This implies $\Pi = \Delta_{12}$. Given this, (i) reduces to $\text{co}(\varphi(P) \cup \varphi(Q)) = \varphi(\text{co}(P \cup Q))$ for all $P, Q \in \mathcal{P}$. Now, assume there is $P \in \mathcal{P}$ such that $\varphi(P) \neq P$. Then, since $\varphi(P) \subseteq P$, there exists $p \in P$ such that $p \notin \varphi(P)$. But then (i) implies $\text{co}(\varphi(P) \cup \{p\}) = \varphi(\text{co}(P \cup \{p\})) = \varphi(P)$, a contradiction.

References

Cabantous L (2007) Ambiguity aversion in the field of insurance: insurers' attitude to imprecise and conflicting probability estimates. *Theory Decis* 62:219–240

Cabantous L, Hilton D, Kunreuther H, Michel-Kerjan E (2011) Is imprecise knowledge better than conflicting expertise? Evidence from insurers' decisions in the United States. *J Risk Uncertain* 1–22

Chambers C, Echenique F (2009) When does aggregation reduce risk aversion? mimeo

Chateauneuf A (1991) On the use of capacities in modeling uncertainty aversion and risk aversion. *J Math Econ* 20:343–369

Clemen R, Winkler R (2007) Aggregating probability distributions. *Advances in decision analysis: from foundations to applications*, pp 154–176

Cooke RM (1991) *Experts in uncertainty: opinion and subjective probability in science*. Oxford University Press, New York

Cooke WE (1906) Forecasts and verifications in Western Australia. *Monthly Weather Rev* 34(1):23–24

Crés H, Gilboa I, Vieille N (2011) Aggregation of multiple prior opinions. *J Econ Theory* 146(6):2563–2582

Debreu G (1954) Representation of a preference ordering by a numerical function. In: Thrall RM, Coombs CH, Davis RL (eds) *Decision processes*. Wiley, New York, pp 159–165

Gajdos T, Tallon J-M, Vergnaud J-C (2004) Decision making with imprecise probabilistic information. *J Math Econ* 40:647–681

- Gajdos T, Hayashi T, Tallon J-M, Vergnaud J-C (2008a) Attitude toward imprecise information. *J Econ Theory* 140(1):27–65
- Gajdos T, Tallon J-M, Vergnaud J-C (2008b) Representation and aggregation of preferences under uncertainty. *J Econ Theory* 141(1):68–99
- Genest C, Zidek J (1986) Combining probability distributions: a critique and an annotated bibliography. *Stat Sci* 1:114–148
- Gilboa I, Schmeidler D (1989) Maximin expected utility with a non-unique prior. *J Math Econ* 18:141–153
- Gilboa I, Samet D, Schmeidler D (2004) Utilitarian aggregation of beliefs and tastes. *J Polit Econ* 112:932–938
- Harsanyi J (1955) Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. *J Polit Econ* 63:309–321
- Kriegler E, Hall J, Held H, Dawson R, Schellnhuber H (2009) Imprecise probability assessment of tipping points in the climate system. *Proc Nat Acad Sci* 106(13):5041–5046
- McConway K (1981) Marginalization and the linear opinion pools. *J Am Stat Assoc* 76(374):410–414
- Mongin P (1995) Consistent Bayesian aggregation. *J Econ Theory* 66:313–351
- Morris P (1974) Decision analysis expert use. *Manag Sci* 1233–1241
- Morris P (1977) Combining expert judgments: a Bayesian approach. *Manag Sci* 679–693
- Nascimento L (2012) The ex-ante aggregation of opinions under uncertainty. *Theor Econ*
- Nau R (2002) The aggregation of imprecise probabilities. *J Stat Plan Inference* 105(1):265–282
- Rennie D (1981) Consensus statements. *New Engl J Med* 304(11):665–666
- Savage L (1954) *The foundations of statistics*. Wiley, New York
- Smithson M (1999) Conflict aversion: preference for ambiguity vs conflict ins sources and evidence. *Org Beha Hum Decis Process* 79:179–198
- Stone M (1961) The opinion pool. *Ann Math Stat* 32:1339–1342
- Troffaes M (2006) Generalizing the conjunction rule for aggregating conflicting expert opinions. *Int J Intel Syst* 21(3):361–380
- Wagner C (1989) Consensus for belief functions and related uncertainty measures. *Theory Decis* 26:295–304